MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. IX.

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WITH MANY

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Mathematical Papers, &c.

Solved Questions.

Unsolved Questions.

2392. (Proposed by Professor CAYLEY.) — It is required, by a real or imaginary linear transformation, to express the equation of a *given* cubic curve in the form

$$xy - z^2 = \sqrt{\left\{(z^2 - x^2)(z^2 - k^2x^2)\right\}}.$$

2394. (Proposed by Professor CREMONA.) — Les plans qui coupent en quatre points harmoniques une courbe gauche de 4e ordre et 2e espèce (courbe gauche unicursale de 4e ordre, sans points doubles) enveloppent une surface de STEINER [see Crelle's Journal, vols. 63 and 64] pour laquelle la courbe donnée est asymptotique.

2446. (Proposed by W. K. CLIFFORD, B.A.)—PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Show that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence show by inversion that, if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

2455. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—If a, b, c be the three sides of a spherical triangle, and k the radius of its polar circle, prove the formula

$$\tan^2 k = \frac{(\cos b \cos c \sec a - 1)(\cos c \cos a \sec b - 1)(\cos a \cos b \sec c - 1)}{4 \sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

2505. (Proposed by the late Rev. R. H. WRIGHT, M.A.)—Prove that

$$\frac{(1+\frac{1}{4}x)^2}{(1+\frac{1}{3}x)(1+\frac{1}{5}x)} \cdot \frac{(1+\frac{1}{8}x)^2}{(1+\frac{1}{7}x)(1+\frac{1}{9}x)} \cdot \text{ad inf.} = \frac{2^{-\frac{3}{2}x}\,\Gamma(x+2)}{\Gamma(\frac{1}{2}x+1)\,.\,\Gamma(\frac{1}{4}x+1)}.$$

2531. (Proposed by Professor SYLVESTER.)—1. Let E denote the event of three points forming the apices of an acute angle triangle. Let p be the probability of E for three points taken anywhere inside a given triangle A; t the probability of E when the three points lie respectively anywhere upon the three sides of A; q, r, s the several probabilities of E when one point is taken at an angle of A, a second anywhere in the side opposite that angle, and a third anywhere inside A. Prove that $p = \frac{1}{5}q + \frac{1}{5}r + \frac{1}{5}s + \frac{2}{5}t$.

2. Obtain an analogous theorem for decomposing the probability of E when the original range of the three points is a tetrahedron.

3. Also give the corresponding theorems of decomposition when E denotes the event of a group of n points satisfying any intrinsic condition of form, and the original range of the group is a triangle, parallelogram, tetrahedron, or parallelepiped.

2555. (Proposed by A. DE MORGAN, F.R.A.S.)—The following is a theorem of which an elementary proof is desired. It was known before I gave it in a totally different form in a communication (April, 1867) to the Mathematical Society on the *conic octagram;* and the

present form is as distinct from the other two as they are from one another. If I, II, III, IV be the consecutive chord-lines of one tetragon inscribed in a conic, and 1, 2, 3, 4 of another; the eight points of intersection of I with 2 and 4, II with 1 and 3, III with 2 and 4, IV with 1 and 3, lie in one conic section. A proof is especially asked for when the first conic is a pair of straight lines. There is, of course, another set of eight points in another conic, when the pairs 13, 24 are interchanged in the enunciation.

2564. (Proposed by M. Collins, B.A.)—A being a curve whose equation is given in the usual Cartesian rectangular coordinates, B the evolute of A, and C the evolute of B; required a general differential expression for the radius of curvature of C, on the usual supposition of dx being taken constant, and likewise on the supposition of dx^2+dy^2 $(=ds^2)$ being taken constant.

2565. (Proposed by M. W. Crofton, B.A.)—1. A convex boundary of any form, of length L, encloses an area Ω. If two straight lines are drawn at random to intersect it, the probability of their intersection lying within it is $p = 2\pi\Omega L^{-2}$.

2. The probability of their intersection lying within any given area ω, which is enclosed within Ω, is $p=2\pi\omega L^{-2}$. [Mr. Crofton remarks that an interesting but much more difficult problem (a general solution to which is probably impracticable) is to find the chance of their intersection lying on a given area ω, *external* to Ω.]

3. If an infinity of random lines are drawn across the given area Ω, their intersections form an assemblage of points covering the plane, the density of which is clearly uniform *within* Ω. Show that, at any *external* point P, the density varies as $\theta-\sin\theta$, where θ is the angle Ω subtends at P.

4. If Ω be any plane area, enclosed by a convex boundary of length L, and θ be the angle it subtends at any external point $P(x, y)$, prove that $\iint(\theta-\sin\theta)\,dx\,dy = \frac{1}{2}L^2-\pi\Omega$, the integral extending over the whole external surface of the plane. [We shall be glad to receive a separate investigation of this theorem, independent of that which follows at once from the preceding theory.]

2574. (Proposed by W. S. B. Woolhouse, F.R.A.S.) — A given straight line is divided at random into n portions, and these are arranged in the order of magnitude; determine the average value of the portion which stands the mth in order.

2580. (Proposed by A. W. Panton, B.A.)—1. If F and F′ are the foci of an oval of Cassini, and C its centre; prove the following construction for the second pair of foci. The circle through F or F′ and the two points where any line through C meets either oval (the curve consisting of two distinct ovals) cuts the axis in one of the required points.

2. If S and S′ be the foci thus found, and P any point on the curve, prove that PS . PS′ is proportional to PC^2.

2597. (Proposed by W. H. H. Hudson, M.A.) — A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: prove that the whole pressure upon its surface is $\frac{1}{3}\pi\rho gh^3\dfrac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\dfrac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}$, where h and 2α

are the height and vertical angle of the cone, and θ is determined from $3\sin 2\theta = 4\sin 2(\theta-\alpha)$.

2612. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If three points P, Q, R be taken at random, one on each side of a triangle ABC, show that the respective chances of the triangle PQR being (1) greater than one-fourth, and (2) greater than one-half, the area of the triangle ABC, are $p_1 = \frac{1}{4}+\frac{1}{2}\log 2$, $p_2 = \frac{1}{4}+\frac{1}{12}\pi-\log 2$.

2616. (Proposed by C. W. MERRIFIELD, F.R.S.)—Let two intersecting tetrahedra have all their edges bisected by the same system of Cartesian axes, each axis through two opposite edges of each tetrahedron; then the solid about the origin has the origin for its centre of figure.

2618. (Proposed by the Rev. N. M. FERRERS, M.A.)—A pack of n different cards is laid, face downwards, on a table. A person names a certain card; that and all the cards above it are shown to him and removed; he names another, and the process is repeated: prove that the chance of his naming the top card during the operation is

$$\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\ldots+\frac{(-1)^n}{\lfloor n}.$$

2619. (Proposed by the EDITOR.)—Trace the form, and find the equation, length, and area of the first negative focal pedal of the ellipse, and deduce therefrom the same for the parabola. (See Solution of Question 1390, *Reprint*, Vol. I., p. 23.)

2621. (Proposed by the Rev. M. M. U. WILKINSON.)—If four points be taken at random on the surface of a sphere, show that the chance of their all lying on some one hemisphere is $\frac{7}{8}$.

2633. (Proposed by Professor TAIT.)—Show that the greatest amount of mechanical effect which can be obtained from a system of equal and similar masses (whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures.

2634. (Proposed by A. CRUM BROWN, D.Sc.)—$4m+2n$ separate strings, having each a black and a white end, are taken. $4m$ of these are united in groups of four by their white ends, and the white ends of all the others are to be left free. In how many ways may the black ends of the system be united two and two so as to form a continuous aggregate? It is of course contemplated that two black ends belonging to the same group of four may be united. This gives rise to the further question:—Divide the above arrangements into classes according to the number of groups of four, each of which has two of its black ends united.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2475. (Proposed by the EDITOR.)—Trace the curve whose equation in triangular coordinates (x, y, z) is

$$\left(\frac{1-x}{x}\right)^{x(1-x)}\left(\frac{1-y}{y}\right)^{y(1-y)}\left(\frac{1-z}{z}\right)^{z(1-z)} = \epsilon^{p};$$

that is, the curve which is the locus of P in Question 2359. (*Reprint*, Vol. VIII., p. 53.)

Solution by W. S. B. WOOLHOUSE, F.R.A.S.

As the triangular coordinates are not altered in value by orthographic projection, no generality will be lost if we consider the question in relation to an equilateral triangle. Also if the perpendicular of the equilateral triangle of reference be unity, the triangular coordinates will become the same as trilinear. And as the proposed equation is symmetrical, the form of the curve will be symmetrical with respect to the equal sides of the triangle of reference. If we put

$$\phi(x) = x(1-x)\log\frac{1-x}{x},$$

the logarithm of the proposed equation is

$$\phi(x)+\phi(y)+\phi(z) = p \text{ ...... } (a).$$

The curve may be effectually traced, and its properties practically developed, by first tabulating the function ϕx for the several values of x, and a specimen of such table is here given. It will be observed that

$$\phi(1-x) = -\phi(x),$$

and that the function is negative for the values of x given in the right-hand column. The numerical value of ϕx is most conveniently obtained by using common logarithms and employing the formula

$$\log\phi(x) = \log\left\{\log(1-x)-\log x\right\} + \left\{\log(1-x)+\log x\right\} + 0{\cdot}3622156.$$

x	$\phi(x)$	x	x	$\phi(x)$	x	x	$\phi(x)$	x
·00	+·00000 −	1·00	·17	+·22373 −	·83	·34	+·14884 −	·66
·01	·04549	·99	·18	·22381	·82	·35	·14083	·65
·02	·07628	·98	·19	·22316	·81	·36	·13256	·64
·03	·10115	·97	·20	·22181	·80	·37	·12406	·63
·04	·12204	·96	·21	·21981	·79	·38	·11534	·62
·05	·13986	·95	·22	·21719	·78	·39	·10642	·61
·06	·15519	·94	·23	·21399	·77	·40	·09731	·60
·07	·16839	·93	·24	·21025	·76	·41	·08804	·59
·08	·17976	·92	·25	·20599	·75	·42	·07863	·58
·09	·18949	·91	·26	·20124	·74	·43	·06908	·57
·10	·19775	·90	·27	·19604	·73	·44	·05942	·56
·11	·20468	·89	·28	·19040	·72	·45	·04967	·55
·12	·21040	·88	·29	·18436	·71	·46	·03983	·54
·13	·21500	·87	·30	·17793	·70	·47	·02993	·53
·14	·21856	·86	·31	·17114	·69	·48	·01998	·52
·15	·22116	·85	·32	·16402	·68	·49	·01000	·51
·16	+·22287 −	·84	·33	+·15658 −	·67	·50	+·00000 −	·50

As an example of the calculation of positions generally, I have taken the case of $p = ·40$, and by means of the table of $\phi(x)$ and equation (α) deduced the following series of points in this particular curve, the perpendicular of the triangle being 1.

$p = ·40$.			Rectangular Coordinate $y' = \frac{y \backsim z}{\sqrt{3}}$.
x	y	z	
·19	·33127	·47873	$\pm$·08514
·20	·30081	·49919	·11454
·22	·26324	·51676	·14637
·24	·23916	·52084	·16263
·26	·22229	·51771	·17056
·28	·20974	·51026	·17350
·30	·20032	·49968	·17284
·32	·19327	·48673	·16943
·34	·18792	·47208	·16406
·36	·18413	·45587	·15689
·38	·18159	·43841	·14828
·40	·18028	·41972	·13824
·42	·18030	·39970	·12667
·44	·18177	·37823	·11343
·46	·18500	·35500	·09815
·48	·19046	·32954	·08030
·50	·20059	·29941	·05705
·51	·20948	·28052	$\pm$·04101

The peculiarities of the curves for different values of p may, however, be more simply inferred from certain special points which we proceed to notice.

With respect to the curve, it is obvious that the three lines which bisect

the opposite sides of the triangle are each of them diametral. We may first find the two points in which the curve meets one of these diameters; to effect which we have to make $y = z = \frac{1}{2}(1-x)$; so that the equation (a) for determining the two values of x is

$$2\phi(x) + \phi\left(\frac{1-x}{2}\right) = p.$$

With the use of the preceding table for $\phi(x)$, we hence easily calculate the following values according to different values of p, viz.:—

$p = \cdot 10$, $x' = \cdot 0211$ and $\cdot 8800$,
$p = \cdot 20$, $x' = \cdot 0540$ „ $\cdot 7715$,
$p = \cdot 30$, $x' = \cdot 1017$ „ $\cdot 6583$,
$p = \cdot 40$, $x' = \cdot 1801$ „ $\cdot 5209$,
$p = \cdot 42$, $x' = \cdot 2050$ „ $\cdot 4848$,
$p = \cdot 44$, $x' = \cdot 2382$ „ $\cdot 4405$,
$p = \cdot 45$, $x' = \cdot 2619$ „ $\cdot 4114$,
$p = \cdot 46$, $x' = \cdot 3032$ „ $\cdot 3646$,

and $p = \frac{2}{3}\log 2 = \cdot 4621$ is the maximum value, and belongs to the centre of gravity of the triangle. By symmetry these values equally determine two points in each diameter, making six axial points of the curve.

We may next find the positions of the points which lie in a straight line drawn through the centre parallel to a side of the triangle. For points on a line so drawn the following values of p for assumed values of z are obtained by direct calculation:—

$3x = 1$, $3z =$	$p =$	diff.	$3x = 1$, $3z =$	$p =$	diff.
0·00	·00000	+8003	0·50	·37757	+1678
·05	·08003	5559	·55	·39435	1472
·10	·13562	4588	·60	·40907	1276
·15	·18150	3943	·65	·42183	1089
·20	·22093	3449	·70	·43272	908
·25	·25542	3048	·75	·44180	736
·30	·28590	2708	·80	·44916	569
·35	·31298	2412	·85	·45485	403
·40	·33710	2144	·90	·45888	241
·45	·35854	+1903	·95	·46129	+81
·50	·37757		1·00	·46210	

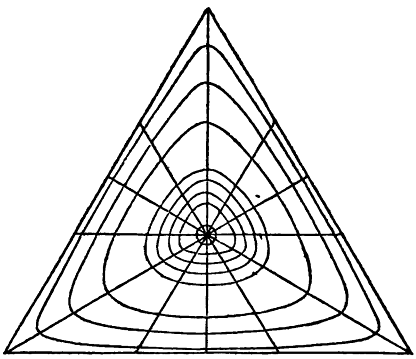
With the use of a table of this kind, extended to each hundredth of value of $3z$, it is easy to directly deduce the value of $3z$ for any given value of p. Then if ρ denote the distance of the point from the centre, the perpendicular of the triangle being unity, we shall have

$$\rho = \frac{2}{3\sqrt{3}}(1-3z) = [9{\cdot}58535]\,(1-3z).$$

The three pairs of points situated on the three lines drawn respectively parallel to the sides are all of them at this same distance from the centre, and are evidently the corners of a regular hexagon. Also if these six points be taken with the six before found, at the distances $x' \backsim \frac{1}{3}$ from the centre, we shall have twelve points of the curve distributed at equal angles round the centre, and these will obviously present a sufficient indication of the particular form of the curve.

If we take as examples the probabilities before assumed, the calculated distances of the twelve special points from the centre are here tabulated.

Prob. p.	Central distances of points		
	on parallel.	towards an angle.	towards a side.
·00	·3849	·6667	·3333
·10	·3591	·5467	·3122
·20	·3184	·4382	·2790
·30	·2596	·3250	·2315
·40	·1661	·1875	·1532
·42	·1375	·1514	·1280
·44	·1002	·1072	·0950
·45	·0744	·0776	·0712
·46	·0306	·0307	·0294



By drawing the diameters and parallels, and laying down the sets of twelve points according to these distances, and then tracing a continuous line through each set of points, the several curves are delineated as shown in the annexed diagram.

It is evident that a similar process is applicable to any curve the trilinear equation of which is symmetrical. In the present case the curve is always convex outwards. For the higher probabilities, near the centre, it is nearly circular; but as the probability becomes less and the curve expands, it becomes partially flattened towards the three sides of the triangle, as if it were in imitation of those sides which, in fact, the curve ultimately becomes when p vanishes.

We may add that it is hence easy to conceive what will become of the several parts of the figure when it is orthogonally projected so as to transform the equilateral triangle into another of any given form.

Note on Question 2471. By Professor Cayley.

In the singularly beautiful solution which Mr. Woolhouse has given of this question, (see *Reprint*, Vol. VIII., p. 100) it is important to note what is the analytical problem solved, and how the solution is obtained. Considering a plane area bounded by any closed convex curve, and in it three

points P, P′, P″, Mr. WOOLHOUSE in vestigates the average area of the triangle PP′P″, viz., this depends on the sextuple integral

$$\int \pm \left\{x'y' - x''y' + x''y - xy'' + xy' - x'y\right\} dx\, dy\, dx'dy'dx''dy'',$$

where the sign $\pm$ has to be taken so that $\pm \left\{\,\right\}$ shall be positive, and where the integration in respect to each set of coordinates extends over the entire closed area; the difficulty is as to the mode of dealing with the discontinuous sign. It is remarked that the integral is

$$=6 \int \pm \left\{x'y'' - x''y' + x''y - xy'' + xy' - x'y\right\} dx\, dy\, dx'dy'dx''dy'' ;$$

the variables in this last expression being restricted in such wise that x, x'', x' are in the order of increasing magnitude; the term $\pm \left\{\,\right\}$ is of the form

$$\pm (x'-x)(y''-\beta)$$

where β is independent of y, and where (as is easily seen) if v'', u'', be the upper and lower ordinate corresponding to the abscissa x'', then β lies between the values u'' and v''. But $x'-x$ is positive, hence the sign $\pm$ must be so taken that $\pm(y''-\beta)$ shall be positive, that is, from $y''=u''$ to $y''=\beta$ the sign is $-$, and from $y''=\beta$ to $y''=v''$ the sign is $+$.

Whence for the integration in regard of y'' we have

$$\int \pm (y''-\beta)\, dy'' = \int_{\beta}^{v''} + (y''-\beta)\, dy'' + \int_{u'}^{\beta} - (y''-\beta)\, dy''$$

$$= \tfrac{1}{2}(v''-\beta)^2 + \tfrac{1}{2}(\beta-u'')^2;$$

and the discontinuous sign $\pm$ is thus got rid of. The remaining integrations are then effected in the order x'', y', y, x', x, the limits being for x'' from x to x', for y' from u' to v', and for y from u to v (if the upper and lower ordinates corresponding to the abscissa x and x' are v, u and v', u' respectively) and finally for x' from x to the maximum abscissa, and for x from the minimum to the maximum abscissa. The final result involves only single definite integrals between the extreme values of x, the functions under the integral sign containing indefinite integrations from the same arbitrary inferior limit, say $x=0$; the form of the result (previous to its simplification by taking the axes to be principal axes through the centre of gravity of the area) is however somewhat complicated; and it would not be easy to show *a posteriori*, that the value is invariantive, that is, independent of the position of the axes: that this is so is of course apparent from the original form of the integral.

2491. (Proposed by the Rev. R. HARLEY, F.R.S.)—Solve, by definite integration, the equation

$$(x^3-1)\frac{d^3y}{dx^3} - 3ax^2\frac{d^2y}{dx^2} + 3a(a+1)\,x\frac{dy}{dx} - a(a+1)(a+2)\,y=0.$$

Solution by the REV. J. L. KITCHIN.

1. Put $x = \epsilon^t$, and D for $\dfrac{d}{dt}$; then the equation becomes

$$\epsilon^{3\theta}(D-a)(D-a-1)(D-a-2)\,y - D(D-1)(D-2)\,y = 0,$$

therefore $y - \frac{D-a-3}{D}\epsilon^{\theta} \cdot \frac{D-a-3}{D}\epsilon^{\theta} \cdot \frac{D-a-3}{D}\epsilon^{\theta} \cdot y = 0,$

or $$\left\{1-\left(\frac{D-a-3}{D}\epsilon^{\theta}\right)^3\right\} y = 0.$$

The solution of this depends on the roots of $1-\rho^3 = 0$. Let these roots be $1, \omega_1, \omega_2$; then

$$\frac{1}{1-\rho^3} = \frac{1}{3(1-\rho)} - \frac{\omega_1}{\omega_2(1-\omega_1)^2} \cdot \frac{1}{\omega_1-\rho} - \frac{\omega_2}{\omega_1(1-\omega_2)^2} \cdot \frac{1}{\omega_2-\rho}.$$

So that if y_1, &c., denote the values of y obtained from these forms separately their sum will be the value of y.

Now $$y_1 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_1 = 0 \text{ gives } y_1 = C_1(x-1)^{a+2}$$

$$\omega_1 y_2 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_2 = 0 \text{ gives } y_2 = C_2(x-\omega_1)^{a+2}$$

$$\omega_2 y_3 - \frac{D-a-3}{D}\epsilon^{\theta} \cdot y_3 = 0 \text{ gives } y_3 = C_3(x-\omega_2)^{a+2}$$

$$\therefore\ y = \frac{C_1}{3}(x-1)^{a+2} - \frac{\omega_1 C_2 (x-\omega_1)^{a+2}}{\omega_2(1-\omega_1)^2} - \frac{\omega_2 C_3 (x-\omega_2)^{a+2}}{\omega_1(1-\omega_2)^2}.$$

2. This equation forms one of a class whose symbolic expression is

$$y - \frac{\left\{(D-a-n)\epsilon^{\theta}\right\}^n}{[D]^n} y = 0,$$

where the lower line is not exponential.

Its general form, not expressed in symbols of operation, is

$$(x^n-1)\frac{d^n y}{dx^n} - \frac{n}{1}x^{n-1}[a]^1\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{1.2}x^{n-2}[a]^2\frac{d^{n-2}y}{dx^{n-2}}$$

$$-\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}[a]^3\frac{d^{n-3}y}{dx^{n-3}} + \ldots + (-1)^n[a]^n = 0,$$

where $[a]^n$ denotes $a(a+1)(a+2)\ldots(a+n-1)$.

It is plain also that this question can be solved, when, instead of being equated to zero, it is equal to a function of x, as X. The equation will then split up into n equations of the form $\frac{dy}{dx} + Py = \phi$, where P and Q are both functions of x; and its general integral can be found whenever these n equations can be integrated in finite terms.

[The Proposer remarks, that the first member of the given equation may be decomposed into three linear differential factors of the first order and degree, and that the equation may be thus put under the form

$$\left[(x-1)\frac{d}{dx}-a\right]\left[(x-\omega^2)\frac{d}{dx}-(a+1)\right]\left[(x-\omega)\frac{d}{dx}-(a+2)\right]y = 0,$$

where ω is an unreal cube root of unity. From this, by successive integrations, he deduces the following results, viz.,

$$\left[(x-\omega^2)\frac{d}{dx}-(a+1)\right]\left[(x-\omega)\frac{d}{dx}-(a+2)\right]y = \mathrm{C}(x-1)^a,$$

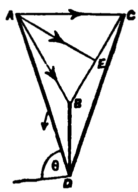
$$\left[(x-\omega)\frac{d}{dx}-(a+2)\right]y = \mathrm{C}'(x-1)^{a+1}+\mathrm{C}''(x-\omega^2)^{a+1},$$

$$y = \mathrm{C}_1(x-1)^{a+2}+\mathrm{C}_2(x-\omega)^{a+2}+\mathrm{C}_3(x-\omega^2)^{a+2}.$$

The final result agrees with Mr. Kitchin's, though it is here expressed in a somewhat simpler form.]

2400. (Proposed by T. T. Wilkinson, F.R.A.S.)—Three equal and heavy rods, in the position of the three edges of an inverted pyramid, are in equilibrium under the following circumstances:—their upper extremities are connected by strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Show that the increase of tension of the strings, corresponding to a given increase of their lengths, varies as $\operatorname{cosec}^3\theta$, where θ is the inclination of each of the rods to the horizontal plane.

Solution by the Proposer; J. Dale; *and others.*



Let ABC be the equilateral triangle formed by the three strings; AD, BD, CD, the three rods; and AE a perpendicular to BC. Put CA $= a$, AD $= 2b$, T = the tension of a string; and W = weight of each rod. Then $\angle$ BAE $= 30°$; and 2T cos 30° = resultant of the tension of AC and AB; which evidently acts in the line AE. This line will also be in the same plane with AD; hence, taking moments round the point D, we have

$$\mathrm{W}b\cos\theta = 2\mathrm{T}\cos 30^\circ\, 2b\sin\theta = 0, \quad \therefore \mathrm{T} = \frac{\mathrm{W}}{2\sqrt{3}}\cot\theta.$$

But AE $= a\cos 30°$, and $\frac{2}{3}$ AE $= 2b\cos\theta$; $\therefore a = 2\sqrt{3}\,.\,b\cos\theta$.

Hence $$\frac{d\mathrm{T}}{da} = \frac{d\mathrm{T}}{d\theta}\cdot\frac{d\theta}{da} = \frac{\mathrm{W}}{12b}\operatorname{cosec}^3\theta,$$ which proves the theorem.

Note on the Solution of Question 2452. By Z.

In the solutions of Question 2452, given on p. 61 of Vol. VIII., of the *Reprint*, it appears to be assumed that if the point Q is not within the partial tetrahedron PABC, it must complete a convex solid PQABC. Passing over the consideration that if QPD happen to be in the same line, Q between P and D is within none of the four partial 4-edra; it

depends upon the way in which we define a convex solid, whether Q external to PABC will always complete a convex solid ABCPQ, or not. The definition of a convex solid, according to Professor Sylvester, is evidently a solid which has no reentrant solid angle. But what if it has a reentrant solid edge? If on the opposite faces of a triangle *abc* we lay two 4-edra *pabc* and *qabc*, so that the faces *pab* and *qab* make a very acute angle, is such a solid *abcpq* convex, when it has the reentrant edge *ab*? If the affirmative is not admitted, the demonstration may have to be reconsidered; for Q external to the 4-edron ABCP will not complete a convex solid ABCPQ in the strict sense, unless the point Q is within one of the solid angles subtended by the three triangles ABP, BCP, CAP. If Q, outside the plane ABP, is in the solid angle subtended by it, a 6-edron is completed by the edges QA, QB, QP. If Q be in the plane ABP, and not in the produced edge PA or PB, a 5-edron is completed having the quadrilateral face APQB. But if Q is not in such a solid angle, but in the dihedral angle, *e.g.* of the planes ABP, BCP, it will be impossible to complete a solid ABCPQ, of which ABC shall be a triangular face, which shall not have the reentrant edge BP. We must draw either the three edges QP, QB, QA, or QP, QB, QC. In the first case QC will be drawable, *i.e.* a line through two angles of the solid ABCPQ, which neither pierces the solid nor is an edge of it; in the other case QA is drawable under the same conditions, a construction impossible in a strictly convex solid.

2478. (Proposed by J. J. WALKER, M.A.)—Deduce the first integrals

$$f+x\left(\frac{\mu}{r}-\frac{dy^2+dx^2}{dt^2}\right)+\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}=0,\ f'\ \&c.\ \&c.$$

(*Méc. Cél.* t. II., c. iii., §. 18) directly from the three equations of motion

$$(1)\ldots\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0,\quad(2)\ldots\frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0,\quad(3)\ldots\frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=0;$$

these first integrals being merely shown by LAPLACE to satisfy the last three differential equations.

Solution by the PROPOSER.

Multiplying (2) by $2x\dfrac{dy}{dt}$ and (3) by $2x\dfrac{dz}{dt}$ and adding, there results;

$$2x\left(\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)+\frac{2\mu x}{r^3}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)=0\ \ldots\ldots\ldots(4).$$

Also we have identically

$$\frac{d}{dt}\left(\frac{\mu x}{r}\right)+\frac{\mu x}{r^3}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-\mu\frac{y^2+z^2}{r^3}\frac{dx}{dt}=0\ \ldots\ldots\ldots\ldots(5),$$

and $$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}-\left(y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right)\frac{dx}{dt}-\frac{ydy+zdz}{dt}\frac{d^2x}{dt^2}=0,$$

which latter equation in virtue of (1) (2) (3) becomes

$$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}+\mu\,\frac{y^2+z^2}{r^3}\,\frac{dx}{dt}+\frac{\mu x}{r^3}\,\frac{ydy+zdz}{dt}=0\,..\,(6).$$

Subtracting (4) from the sum of (5) and (6)

$$\frac{d}{dt}\left(\frac{ydy+zdz}{dt}\cdot\frac{dx}{dt}+\frac{\mu x}{r}\right)-2x\left(\frac{d^2y}{dt^2}\,\frac{dy}{dt}+\frac{d^2z}{dt^2}\,\frac{dz}{dt}\right)-\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)\frac{dx}{dt}=0.$$

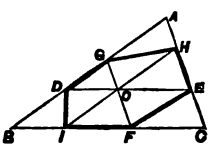
Integrating, f being the arbitrary constant,

$$f+\frac{ydy+zdz}{dt}\,\frac{dx}{dt}+\frac{\mu x}{r}-x\left(\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)=0.$$

By interchanging x and y, x and z successively, we have two other analogous first integrals.

2489. (Proposed by S. WATSON.)—Through a point O, within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB, respectively; find the locus of O when the area of the hexagon DIFEHG is constant.

I. *Solution by* T. DOBSON, B.A.; W. H. LAVERTY; W. CHADWICK; *and others.*



The sum of the parallelograms (OA, OB, OC) at the vertices of the triangle is evidently constant, therefore the sum of the remaining three triangles (OIF, OEH, OGD) is also constant, and $=n^2\Delta$ say.

If (α, β, γ) be the trilinear coordinates of O, the base of the triangle OIF is

$$\text{IF}=\alpha\,(\cot \text{B}+\cot \text{C})=\alpha \sin \text{A} \operatorname{cosec} \text{B} \operatorname{cosec} \text{C}.$$

Hence $\quad \alpha^2\sin^2\text{A}+\beta^2\sin^2\text{B}+\gamma^2\sin^2\text{C}=2n^2\Delta\,.\sin\text{A}\sin\text{B}\sin\text{C},$

or, multiplying by 4R^2, $\quad a^2\alpha^2+b^2\beta^2+c^2\gamma^2=n^2\,(2\Delta)^2$

is the equation to the locus required.

In triangular coordinates this takes the simple form

$$x^2+y^2+z^2=n^2\,................\,(\alpha).$$

[If we put $k=\dfrac{n^2}{1-n^2}=\dfrac{2m-1}{2-2m}$, where $m\Delta$ is the area of the hexagon, the equation (α) may be written

$$x^2+y^2+z^2-2k\,(yz+zx+xy)=0\,........\,(\beta),$$

which shows that the locus of O is an *ellipse*, having its centre at the centre of gravity of the triangle ABC. If the hexagon is three-fourths of the triangle, we have $m=\frac{3}{4}$, $k=1$; and the equation (β) of the locus becomes

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0\,............\,(\gamma),$$

which is that of an ellipse inscribed in the triangle.]

II. *Solution by* R. TUCKER, M.A.

Let $CI = a-x = na$, then $n = \dfrac{a-x}{a}$; and also $IH = nc$, and $CH = nb$;

therefore $CF = (nc-y)\dfrac{a}{c}$, $FO = \dfrac{by}{c}$, $AG = nc-y$, $AH = b(1-n)$.

Then the difference between twice the triangle and twice the hexagon = a constant = $\lambda^2 \sin B$ suppose; hence the equation to the locus of O is

$$xy \sin B + (a-x)y \sin B - \frac{aby^2}{c^2} \sin C + \frac{bx}{a}\left(\frac{a-x}{a}c-y\right) \sin A = \lambda^2 \sin B,$$

or
$$a^2y^2 + acxy + c^2x^2 - a^2cy - ac^2x + \lambda^2 ac = 0,$$

the equation to an ellipse [possessing the properties stated in the note to the foregoing solution.]

2470. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$\frac{n-1}{1} \cdot \frac{2^2}{3} - \frac{(n-1)(n-2)}{1.2} \cdot \frac{2^3}{4} + \&c. = \frac{n+2}{n+1}, \ (n \text{ even}).$$

Solution by SAMUEL ROBERTS, M.A.

We have $\quad x(1-x)^{n-1} = x-(n-1)x^2 + \dfrac{(n-1)(n-2)}{1.2}x^3 - \&c.$

Consequently, if N denote the left hand member of the equality in the question,

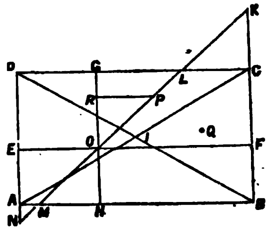
$$N = 1 - \frac{1}{2}\int_0^{-2} x(1-x)^{n-1}\,dx = 1 + \frac{1}{n+1} \ (n \text{ even}).$$

If for the arguments 2^2, 2^3, &c., we write p^2, p^3, &c., the general expression is

$$\frac{p}{2} - \frac{1}{p}\left\{\frac{1-(1-p)^n(1+np)}{n(n+1)}\right\}; \text{ and for } p=2 \ (n \text{ odd}), \text{ we have } N' = \frac{n-1}{n}.$$

2350. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.) — Determine the average area of all triangles drawn on the surface of a given rectangle which have one angle at a given point.

I. *Solution by* STEPHEN WATSON.



Let O be the given point in the rectangle ABCD, through which draw EF, GH, parallel respectively to AB, AD; and draw also NOK meeting AD, AB, CD, CB in N, M, L, K respectively. Take P any point in OL, and draw PR parallel to CD. Put $AB = a$, $BC = b$, $AH = a_1$, $HB = a_2$, $AE = b_1$, $ED = b_2$, $RP = x$, $OR = y$, and $\angle ROP = \phi$. Then when Q lies in the triangles MBK, LCK, LDN, MAN, the sum of the areas of the triangle OPQ is respectively

$$\frac{1}{12xy}(a_2y+b_1x)^3 \ldots (1),\quad \frac{1}{12xy}(a_2y-b_2x)^3 \ldots (2),$$

$$\frac{1}{12xy}(a_1y+b_2x)^3 \ldots (3),\quad \frac{1}{12xy}(a_1y-b_1x)^3 \ldots (4).$$

For $OP = y\sec\phi$, distance of centroid of triangle MBK from $MK = \frac{1}{3}BK\sin\phi$, and area of triangle $MBK = \frac{1}{2}BK^2\tan\phi$; hence

$$\tfrac{1}{2}OP\,.\,\tfrac{1}{3}BK\sin\phi\,.\,\tfrac{1}{2}BK^2\tan\phi = \tfrac{1}{12}\,y\tan^2\phi\,(a_2\cot\phi+b_1)^3 = \frac{1}{12xy}(a_2y+b_1x)^3,$$

which proves (1). Similarly (2), (3), (4) may be established. Since Q must always lie in the rectangle ABCD, (2) and (4) are *subtractive*; but when $x > \dfrac{a_2y}{b_2}$ (say $> x_2$), the triangle LCK is formed on the opposite side of C, is still subtractive, and (2) is *negative*; hence in this case (2) becomes *additive*. Similarly when $x > \dfrac{a_1y}{b_1}$ (say $> x_1$), (4) becomes additive. Consequently when Q takes every position in the rectangle ABCD, and P in OFCG, the sum of the areas of the triangle OPQ is

$$\int_0^{a_2} dy\left\{\int_0^{a_2}[(1)-(2)+(3)-(4)]\,dx + 2\int_{x_2}^{a_2}(2)\,dx + 2\int_{x_1}^{a_2}(4)\,dx\right\}$$

$$= \int_0^{b_2} dy\left\{\frac{y^2}{6}\left(a_1^3\log\frac{a_2b_1}{a_1y} + a_2^3\log\frac{b_2}{y}\right) + \tfrac{1}{3}aa_2^2(b_1^2+b_2^2)\right.$$

$$\left. -\tfrac{1}{4}aa_2(a_1-a_2)(b_1-b_2)\,y + \tfrac{13}{36}(a_1^3+a_2^3)\,y^2\right\}$$

$$= \tfrac{1}{18}a_1^3b_2^3\log\frac{a_2b_1}{a_1b_2} + \tfrac{13}{108}(a_1^3+a_2^3)\,b_2^3 + \tfrac{1}{3}aa_2b_2\left\{ba_2b_1 - a_1b_2(b_1-b_2)\right\} \ldots (5).$$

Under similar conditions, when P lies in the rectangles AHOE, OEDG, OFBH, the sum of the areas is, respectively,

$$\tfrac{1}{18}a_1^3b_2^3\log\frac{a_2b_1}{a_1b_2} + \tfrac{13}{108}(b_1^3+b_2^3)\,a_1^3 + \tfrac{1}{3}ba_1b_1\left\{aa_2b_1 + a_1b_2(a_1-a_2)\right\} \ldots (6),$$

$$\tfrac{1}{18}a_1^3b_1^3\log\frac{a_2b_2}{a_1b_1} + \tfrac{13}{108}(b_1^3+b_2^3)\,a_1^3 + \tfrac{1}{3}ba_1b_2\left\{aa_2b_2 + a_1b_1(a_1-a_2)\right\} \ldots (7),$$

$$\tfrac{1}{18}a_1^3b_1^3\log\frac{a_2b_2}{a_1b_1} + \tfrac{13}{108}(a_1^3+a_2^3)\,b_1^3 + \tfrac{1}{3}aa_2b_1\left\{ba_2b_2 + a_1b_1(b_1-b_2)\right\} \ldots (8).$$

Therefore the required average is

$$\frac{1}{a^2b^2}\left\{(5)+(6)+(7)+(8)\right\}=\frac{1}{a^2b^2}\left\{\frac{a_1^3}{9}\left(b_1^3\log\frac{a_2b_2}{a_1b_1}+b_2^3\log\frac{a_2b_1}{a_1b_2}\right)\right.$$

$$\left.+\tfrac{13}{108}(3a_1^3+a_2^3)(b_1^3+b_2^3)+\tfrac{1}{4}\,aba_1a_2(b_1^2+b_2^2)+\tfrac{1}{4}bb_1b_2(a_1^2-aa_2)(a_1-a_2)\right\}\ (9).$$

The above is on the supposition that O lies above AC and below BD, that is in the triangle AID. When O lies in the triangle AIB the average is

$$\frac{1}{a^2b^2}\left\{\frac{b_1^3}{9}\left(a_1^3\log\frac{a_2b_2}{a_1b_1}+a_2^3\log\frac{a_1b_2}{a_2b_1}\right)+\tfrac{13}{108}\,(a_1^3+a_2^3)\,(3b_1^3+b_2^3)\right.$$

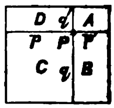
$$\left.+\tfrac{1}{4}\,abb_1b_2\,(a_1^2+a_2^2)+\tfrac{1}{4}\,aa_1a_2\,(b_1^2-bb_2)\,(b_1-b_2)\right\}\ldots\ldots(10).$$

When O lies in the triangle BIC we have merely to interchange a_1, a_2 and b_1, b_2 in (9); and when in the triangle CID the same interchange of letters in (10).

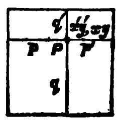
When O lies at I, $a_1=a_2=\frac{1}{2}a$, $b_1=b_2=\frac{1}{2}b$, and the average is $\frac{5}{108}\,ab$.

II. *Solution by the* Proposer.

If the diagram be orthographically projected upon another plane making any assigned angles with that of the given rectangle, it is evident that the area of the triangle will always have the same relative value when compared with the total area. We may therefore simplify the investigation by taking a square, the side of which is unity, as it can be projected into a rectangle or parallelogram of any proposed form.



Let P be the given point, and let it be taken as the origin, with the coordinate axes drawn parallel to the sides of the square, the segments of these axes, intercepted by the sides, being denoted by p, p', q, q', as marked in the annexed diagram. For the sake of brevity, the four coordinate spaces into which the square is divided are designated by the letters A, B, C, D, so as to enable us to distinguish the different localities in which the points are supposed to be taken. We shall also denote the sum of the areas of the triangles by (AA) when the two points Q, R both occupy the space A; by (AB) when one of those points is on A and the other on B; by (AC) when one point is on A and the other on C; and so on. The three cases here stated will determine all others by analogy.



Case I.—Let Q (xy) and R $(x'y')$ be both in A, with R to the left of Q, or $x'<x$.

1. When R is above the side PQ, $\Delta=\dfrac{xy'-x'y}{2}$, and the limits are $y'\left(y\dfrac{x'}{x}\ldots q'\right)$, $x'(o\ldots p')$, &c.

therefore
$$\int\Delta\,dy'=\frac{(xy'-x'y)^2}{4x}=\frac{(q'x-x'y)^2}{4x},$$

$$\iint\Delta\,dx'dy'=-\frac{(q'x-x'y)^3}{12xy}=\frac{q'^3x^3-(q'-y)^3x^3}{12xy},$$

$$=\frac{x^2}{12y}(3q'^2y-3q'y^2+y^3)=\frac{q'^2x^2}{4}-\frac{q'x^2y}{4}+$$

$$\iiint \Delta\, dy\, dx'\, dy' = q'^3 x^2 \left(\tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{36}\right) = \tfrac{11}{72} q'^3 x^2,$$

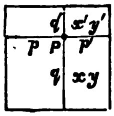
$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \tfrac{11}{216} p'^3 q'^3 \ldots\ldots\ldots\ldots (1).$$

2. When R is below PQ, $\Delta = \frac{1}{2}(x'y - xy')$, and the limits are $y'\left(o \ldots y\dfrac{x'}{x}\right)$, $x'(o \ldots x)$, &c.

therefore $\displaystyle\int \Delta\, dy' = -\frac{(x'y - xy')^2}{4x} = \frac{x'^2 y^2}{4x}$, $\displaystyle\iint \Delta\, dx'\, dy' = \frac{x'^3 y^2}{12x} = \frac{x^2 y^2}{12}$,

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p'^3 q'^3}{108} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

therefore $\qquad (AA) = p'^3 q'^3 \left(\tfrac{11}{216} + \tfrac{1}{108}\right) = \tfrac{13}{216} p'^3 q'^3.$



Case II.—Let Q be in B and R in A; then, changing the sign of y, $\Delta = \frac{1}{2}(xy' + x'y)$, and y', x', y, x take full limits;

therefore $\displaystyle\int \Delta\, dy' = \frac{(xy' + x'y)^2}{4x} = \frac{(q'x + x'y)^2 - x'^2 y^2}{4x}$

$$= \frac{q'^2 x^2 + 2q' x x' y}{4x} = \frac{q'^2 x}{4} + \frac{q' x' y}{2},$$

$$\iint \Delta\, dx'\, dy' = \frac{q'^2 x x'}{4} + \frac{q' y x'^2}{4} = \frac{p' q'^2}{4} x + \frac{p'^2 q'}{4} y,$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p' q'^2}{4} xy + \frac{p'^2 q'}{8} y^2 = \frac{p' q'^2}{4} qx + \frac{p'^2 q'}{8} q^2,$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p' q'^2}{8} qx^2 + \frac{p'^2 q'}{8} q^2 x = \tfrac{1}{8}(p'^3 q'^2 q + p'^3 q' q^2);$$

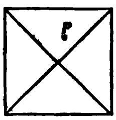
therefore $\qquad (AB) = \dfrac{p'^3}{8} qq'(q + q') - \dfrac{p'^3}{8} qq'.$

Hence, for triangles which lie to the right of P,

$$(AA) + (BB) + (AB) = p'^3 \left\{\tfrac{13}{216}(q^3 + q'^3) + \tfrac{1}{8} qq'(q + q')\right\}$$

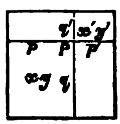
$$= p'^3 \left\{\tfrac{13}{216}(q + q')^3 - \tfrac{1}{18} qq'(q + q')\right\} = p'^3 \left(\tfrac{13}{216} - \tfrac{1}{18} qq'\right),$$

which is therefore the form of expression when P is in the side of a rectangle.



Case III.—Let Q be in C and R in A; and suppose the fixed point P to be situated in the upper diagonal space as in the first of the annexed diagrams; then $pq' < p'q$, or $\dfrac{q'}{p'} < \dfrac{q}{p}$.

1. When R is above PQ, $\Delta = \dfrac{xy' - yx'}{2}$, and we have the limits $y'\left(y\dfrac{x'}{x} \ldots q'\right)$;



therefore $\qquad \displaystyle\int \Delta\, dy' = \frac{(xy' - yx')^2}{4x} = \frac{(q'x - yx')^2}{4x}.$

The other limits are as follows:—

$$\frac{y}{x} < \frac{q'}{p'},\ \Big|\ x'(o \ldots p'), \quad y\left(o \ldots x\frac{q'}{p'}\right),\ x(o \ldots p) \ldots\ldots\ldots (1),$$

$$\frac{y}{x} > \frac{q'}{p'}\ \Big|\ x'\left(o \ldots q'\frac{x}{y}\right),\ y\left(x\frac{q'}{p'} \ldots q\right),\ x(o \ldots p) \ldots\ldots\ldots (2);$$

therefore (1) $\iint \Delta\, dx'\, dy' = -\dfrac{(q'x - yx')^3}{12xy} = \dfrac{q'x^3 - (q'x - yx')^3}{12xy}$

$$= q'x'\frac{q'x - yx'}{4} + \frac{y^3x'^3}{12xy} = p'q'\frac{q'x - p'y}{4} + \frac{p'^3y^2}{12x},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'q'}{4}\left(q'xy - p'\frac{y^2}{2}\right) + \frac{p'^3y^3}{36x} = \frac{p'q'}{4}\left(\frac{q'^2x^2}{p'} - \frac{q'^2x^2}{2p'}\right) + \frac{q'^3x^2}{36}$$

$$= q'^3x^2\left(\tfrac{1}{8} + \tfrac{1}{36}\right) = \tfrac{11}{72}q'^3x^2,$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \tfrac{11}{216}p^3q'^3;$$

$$(2) \iint \Delta\, dx'\, dy' = -\frac{(q'x - yx')^3}{12xy} = \frac{q'^3x^3}{12xy} = \frac{q'^3x^2}{12y},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{q'^3x^2}{12}\log y = \frac{q'^3x^2}{12}\log\frac{p'q}{q'x},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{q'^3x^3}{36}\log\frac{p'q}{q'x} + \frac{q'^3x^3}{108} = p^3q'^3\left(\tfrac{1}{108} + \tfrac{1}{36}\log\frac{p'q}{pq'}\right);$$

therefore $\qquad (1) + (2) = p^3q'^3\left(\tfrac{13}{216} + \tfrac{1}{36}\log\dfrac{p'q}{pq'}\right).$

2. When R is below PQ, $\Delta = \frac{1}{2}(yx' - xy')$, and $y' < y\dfrac{x'}{x}$. The several limits are as follows:—

$$\frac{y}{x} > \frac{q'}{p'}\begin{cases} y\dfrac{x'}{x} > q' & \begin{cases} y'(o \ldots q'), & x'\left(q'\dfrac{x}{y} \ldots p'\right),\ y\left(\dfrac{q'}{p'}x \ldots q\right) \ldots\ldots (1), \\ y'\left(o \ldots y\dfrac{x'}{x}\right), & x'\left(o \ldots q'\dfrac{x}{y}\right), \quad ,, \quad \ldots\ldots (2), \end{cases} \\ y\dfrac{x'}{x} < q' & \end{cases}$$

$$\frac{y}{x} < \frac{q'}{p'} \qquad ,, \qquad ,, \qquad x'(o \ldots p'), \quad y\left(o \ldots \frac{q'}{p'}x\right) \ldots\ldots (3),$$

therefore (1) $\displaystyle\int \Delta\, dy' = -\frac{(yx' - xy')^2}{4x} = \frac{y^2x'^2 - (yx' - q'x)^2}{4x} = \frac{q'x'y}{2} - \frac{q'^2x}{4},$

$$\iint \Delta\, dx'\, dy' = \frac{q'x'^2y}{4} - \frac{q'^2xx'}{4} = \frac{p'^2q'}{4}y - \frac{p'q'^2}{4}x,$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'^2q'}{8}y^2 - \frac{p'q'^2}{4}xy = \frac{p'^2q'q^2}{8} - \frac{p'q'^2qx}{4} + \frac{q'^3x^2}{8},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p'^2q'pq^2}{8} - \frac{p'q'^2p^2q}{8} + \frac{p^3q'^3}{24};$$

$$(2) \int \Delta\, dy' = \frac{y^2x'^2}{4x},$$

$$\iint \Delta\, dx'\, dy' = \frac{y^2x'^3}{12x} = \frac{y^2}{12x}\cdot\frac{q'^3x^3}{y^3} = \frac{q'^3x^2}{12y},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{q'^3x^2}{12}\log y = \frac{q'^3x^2}{12}\log\frac{p'q}{q'x},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = p^3 q'^3 \left(\tfrac{1}{108} + \tfrac{1}{36} \log \frac{p'q}{pq'}\right);$$

$$(3)\ \iint \Delta\, dx'\, dy' = \frac{y^2 x'^3}{12x} = \frac{p'^3 y^2}{12x},$$

$$\iiint \Delta\, dy\, dx'\, dy' = \frac{p'^3 y^3}{36x} = \frac{p'^3}{36x} \cdot \frac{q'^3 x^3}{p'^3} = \frac{q'^3 x^2}{36},$$

$$\iiiint \Delta\, dx\, dy\, dx'\, dy' = \frac{p^3 q'^3}{108};$$

therefore $(1) + (2) + (3) = p^3 q'^3 \left(\tfrac{13}{216} + \tfrac{1}{36} \log \dfrac{p'q}{pq'}\right) + \dfrac{pqp'q'}{8} (p'q - pq')$.

But $\qquad p'q - pq' = p'(1 - q') - pq' = p' - q' = q - p;$

therefore $(1) + (2) + (3) = p^3 q'^3 \left(\tfrac{13}{216} + \tfrac{1}{36} \log \dfrac{p'q}{pq'}\right) + \dfrac{pqp'q'}{8} (q - p)$.

This added to the former value gives

$$(AC) = p^3 q'^3 \left(\tfrac{13}{108} + \tfrac{1}{18} \log \frac{p'q}{pq'}\right) + \frac{pqp'q'}{8} (q - p).$$

Similarly, interchanging p and p',

$$(BD) = p'^3 q'^3 \left(\tfrac{13}{108} + \tfrac{1}{18} \log \frac{pq}{p'q'}\right) + \frac{pqp'q'}{8} (q - p').$$

Also, from what precedes,

$$(AA + BB + AB) = p'^3 \left(\tfrac{13}{216} - \frac{qq'}{18}\right), \quad (AD) = q'^3 \frac{pp'}{8},$$

$$(CD + CC + DD) = p^3 \left(\tfrac{13}{216} - \frac{qq'}{18}\right), \quad (BC) = q^3 \frac{pp'}{8}.$$

These several values will be reproduced when the positions of Q and R are interchanged, and therefore to include all positions they must be doubled. And the collective number of positions of Q and R being each represented by the area of the square, the total number of positions is unity. The sum of the areas is therefore in the present case equal to the average area. That is, collecting the last six values and doubling the result,

$$(\Delta) = \tfrac{13}{108} - \frac{pp'}{9} - \frac{qq'}{9} - \frac{5pp'qq'}{12} + \tfrac{13}{54} q'^3 (1 - 3pp') + \frac{pp'qq'}{4} (q - q')$$
$$+ \frac{q'^3}{9} \left(p^3 \log \frac{p'q}{pq'} + p'^3 \log \frac{pq}{p'q'}\right),$$

which expresses the average area required in parts of the total area of the rectangle.

The transcendentals vanish only when P is in a side, in a diagonal, or at the centre.

With P in a side, $\qquad (\Delta) = \tfrac{13}{108} - \dfrac{pp'}{9}$ (average $\tfrac{11}{108}$),

" at a corner, $\qquad (\Delta) = \tfrac{13}{108}$,

" middle of side, $\qquad (\Delta) = \tfrac{10}{108}$,

" at the centre, $\qquad (\Delta) = \tfrac{5}{108}$.

In reference to the rectangle, p, q, p', q' are the four perpendiculars divided by the sides to which they are respectively parallel. Of these four

fractions, q' is the least in value, and refers to the perpendicular which does not intersect a diagonal; q is the greatest, and involves the intersection of both diagonals. The intermediate values p, p' each alike intersect one diagonal, and with respect to them the formula is symmetrical.

Addendum.—As a check on the calculation, we may suppose P to vary its position on the rectangle, and by integrating the value of (Δ) with respect to p', q', so as to obtain the average area when all three points occupy all positions. To integrate for the right hand half of the upper diagonal space the limits are $q'(o \ldots p')$ and $p'(o \ldots \frac{1}{2})$.

Taking the first six terms of the expression, we have

$$u = \tfrac{13}{108} - \frac{pp'}{9} - \frac{qq'}{9} - \frac{5pp'qq'}{12} + \tfrac{13}{54} q'^3 (1-3pp') + \frac{pp'qq'}{4}(q-q'),$$

$$\int u\,dq'\,(o \ldots p') = \tfrac{13}{108} p' - \frac{pp'^2}{9} - \frac{1}{9}\left(\frac{p'^2}{2} - \frac{p'^3}{3}\right) - \frac{5pp'}{12}\left(\frac{p'^2}{2} - \frac{p'^3}{3}\right)$$

$$+ \tfrac{13}{216} p'^4 (1-3pp') + \frac{pp'}{4}\left(\frac{p'^2}{2} - p'^3 + \frac{p'^4}{2}\right)$$

$$= \tfrac{13}{108} p' - \tfrac{1}{8} p'^2 + \tfrac{7}{108} p'^3 + \tfrac{7}{216} p'^4 + \tfrac{1}{18} p'^5 + \tfrac{1}{18} p'^6.$$

As the point P traverses only $\frac{1}{8}$ of the total area, multiply by 8, and we get

$$8\iint u\,dp'\,dq'\,(o \ldots \tfrac{1}{2}) = 8\left(\tfrac{13}{216}\cdot\tfrac{1}{4} - \tfrac{1}{18}\cdot\tfrac{1}{8} + \tfrac{7}{432}\cdot\tfrac{1}{16} + \tfrac{7}{1080}\cdot\tfrac{1}{32}\right.$$

$$\left. + \tfrac{1}{108}\cdot\tfrac{1}{64} + \tfrac{1}{126}\cdot\tfrac{1}{128}\right) = \tfrac{8}{105}.$$

For the two remaining terms

$$u_1 = \frac{q'^3}{9} p^3 \log \frac{p'q}{pq'},$$

$$\int u_1\,dq' = \frac{q'^4}{36} p^3 \log \frac{p'q}{pq'} + \frac{p^3}{36} \int q'^4 \left(\frac{dq'}{1-q'} + \frac{dq'}{q'}\right)$$

$$= \frac{q'^4}{36} p^3 \log \frac{p'q}{pq'} + \frac{p^3}{36} \int \frac{q'^3\,dq'}{1-q'}$$

$$\text{(between limits)} = \frac{(1-p')^3}{36} \int \frac{p'^3\,dp'}{1-p'};$$

$$\text{therefore } 8\iint u_1\,dp'\,dq' = -\frac{(1-p')^4}{18} \int \frac{p'^3\,dp'}{1-p'} + \tfrac{1}{18} \int p'^3 (1-p')^3\,dp'$$

$$= -\frac{(1-p')^4}{18}\left(\log \frac{1}{1-p'} - p' - \frac{p'^2}{2} - \frac{p'^3}{3}\right) + \tfrac{1}{18}\left(\frac{p'^4}{4} - \tfrac{3}{5} p'^5 + \tfrac{1}{2} p'^6 - \frac{p'^7}{7}\right)$$

$$= \frac{1}{18 . 16}\left\{-\log 2 + \tfrac{1}{3} + \tfrac{1}{8} + \tfrac{1}{24} + \left(\tfrac{1}{4} - \tfrac{3}{10} + \tfrac{1}{8} - \tfrac{1}{56}\right)\right\} = \tfrac{1}{288}\left(-\log 2 + \tfrac{76}{105}\right)$$

$$\ldots\ldots\ldots\ldots (\alpha).$$

$$u_2 = \frac{q'^3}{9} p'^3 \log \frac{pq}{p'q'},$$

$$\int u_2\,dq' = \frac{q'^4}{36} p'^3 \log \frac{pq}{p'q'} + \frac{p'^3}{36} \int \frac{q'^3\,dq'}{1-q'}$$

$$\text{(limits)} = \frac{p'^7}{18} \log \frac{p}{p'} + \frac{p'^3}{36} \int \frac{p'^3\,dp'}{1-p'},$$

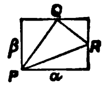
$$8\iint u_2\,dp'\,dq' = \frac{p'^8}{18} \log \frac{p}{p'} + \tfrac{1}{18} \int \frac{p'^7\,dp'}{1-p'} + \frac{p'^4}{18} \int \frac{p'^3\,dp'}{1-p'} - \tfrac{1}{18} \int \frac{p'^7\,dp'}{1-p'}$$

$$= \frac{p'^3}{18}\log\frac{p}{p'} + \frac{p'^4}{18}\left(\log\frac{1}{1-p'} - p' - \frac{p'^2}{2} - \frac{p'^3}{3}\right)$$

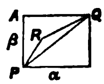
$$\text{(limits)} = \frac{1}{18.16}\left(\log 2 - \tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{24}\right) = \tfrac{1}{288}\left(\log 2 - \tfrac{2}{3}\right) \ldots\ldots\ldots\ldots\ldots (\beta).$$

therefore $\qquad (\alpha)+(\beta) = \frac{1}{288}\left(\frac{76}{105} - \frac{2}{3}\right) = \frac{1}{5040}.$

The value of (Δ) for the whole surface is therefore $\frac{8}{105} + \frac{1}{5040} = \frac{11}{144}$, which agrees with the result of other processes of a more direct and simple kind. Perhaps the following is as easy as any.



Let the triangle PQR be conceived to be circumscribed by the least rectangle having its sides (α, β) parallel to the sides of the square. Then the configurations of the points with reference to this rectangle will present two cases.



Case 1.—One point P at a corner and the others in the remote sides. The average triangle has the latter points in the bisections of the sides, and its area is $\frac{3}{8}\alpha\beta$. The positions of $P = 4$, one for each corner; positions of $Q, R = \alpha\beta$; and permutations of $P, Q, R = 6$, giving in all $24\alpha\beta$ positions; and therefore areas $= \frac{3}{8}\alpha\beta \times 24\alpha\beta = 9\alpha^2\beta^2$.

Case 2.—Two points at opposite corners and the third anywhere on the surface. Here the average triangle evidently has R at the centre of gravity of APQ, and its area is $\frac{1}{6}\alpha\beta$. The positions of $P, Q = 2$; of $R = \alpha\beta$; and permutations $= 6$, giving $12\alpha\beta$ positions, and areas $= \frac{1}{6}\alpha\beta \times 12\alpha\beta = 2\alpha^2\beta^2$.

Therefore in respect of the rectangle $(\alpha\beta)$ the sum of the areas of the triangles $= 9\alpha^2\beta^2 + 2\alpha^2\beta^2 = 11\alpha^2\beta^2$.

Now the number of positions of the rectangle $(\alpha\beta)$ upon the surface of the given square, the side of which is unity, is evidently $(1-\alpha)(1-\beta)$. Therefore when P, Q, R range over the whole surface, we obtain

$$(\Delta) = \iint 11\alpha^2\beta^2 d\alpha d\beta\,(1-\alpha)(1-\beta) = \tfrac{11}{144}.$$

The same result is also readily found from the formula stated in Question 2471 (*Reprint*, Vol. VIII., p. 100) to the solution of which it is appended amongst other examples.

2484. (Proposed by J. Griffiths, M.A.)—If A, B, C denote the angles of a plane triangle, L, M, N known quantities, and $f(x) \equiv x + x^{-1}$, find the value of k obtained by eliminating x and y from the equations

$$f(x) + 2\cos A = kL, \quad f(y) + 2\cos B = kM, \quad f(xy) + 2\cos C = kN.$$

Solution by W. H. Laverty; R. Tucker, M.A.; W. Chadwick; *and many others.*

Let $kL - 2\cos A = x + \dfrac{1}{x} = 2\cos\theta$, and $kM - 2\cos B = y + \dfrac{1}{y} = 2\cos\phi$,

then $\qquad kN - 3\cos C = xy + \dfrac{1}{xy} = 2\cos(\theta+\phi)$;

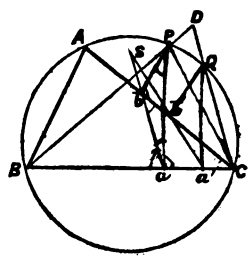
therefore $\left\{(kN-2\cos C)-2\cos\theta\cos\phi\right\}^2 = 4\sin^2\theta\sin^2\phi,$

therefore $(kN-2\cos C)^2-(kL-2\cos A)(kM-2\cos B)(kN-2\cos C)$
$$= 4-(kL-2\cos A)^2-(kM-2\cos B)^2,$$
which gives a cubic in k, the term not involving k being
$$8\cos A\cos B\cos C-4+4(\cos^2 A+\cos^2 B+\cos^2 C),$$
which is equal to zero, since $A+B+C=180°$. Hence one value of k is $k=0$, and the other two values may be found from the quadratic equation
$$k^2 LMN-k(\Sigma L^2+2\Sigma MN\cos A)+4\Sigma L\sin B\sin C=0.$$

2480. (Proposed by R. TUCKER, M.A.)—A circle is drawn round a triangle ABC; and, from any point D, lines DB, DC are drawn cutting the circle in P and Q, whose feet-perpendicular lines with reference to ABC intersect in S; prove that the angle S is equal to the difference between the angles A and D.

Solution by T. DOBSON, B.A.; W. CHADWICK; *and others.*

Draw Pa, Qa', and Pb, Qb', perpendicular to BC, CA respectively; and produce ab, $a'b'$ to meet in S. Then the points abPC lie in a circle, as do also the points $a'b'$QC. Hence $\angle S = baB - b'a'B = bPC - b'QC = PCQ = BPC - D = A - D$. If D be within the circle, $S = D - A$.



COR.—If D′, D″ be the angles subtended by AC and AB at the point D, and S′, S″ angles analogous to S; we have
$$D+D'+D''\pm(S+S'+S'')=\pi.$$

2449 (From the TRINITY COLLEGE EXAMINATION PAPERS for 1865.)—If $\dfrac{f(x)}{F(x)}$, where $f(x)$, $F(x)$ are quadratic expressions in x, be expanded in ascending powers of x, prove that no term can vanish unless $\log\dfrac{f(\alpha)}{f(\beta)}+\log\left(\dfrac{\alpha}{\beta}\right)$ be a positive integer, α and β being the roots of $F(x)$, supposed real and unequal.

Solution by J. McDowell, M.A., F.R.A.S.

Decomposing into partial fractions, we have

$$\frac{f(x)}{F(x)} = \frac{f(x)}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\left\{\frac{f(\alpha)}{x-\alpha} - \frac{f(\beta)}{x-\beta}\right\}$$

$$= \frac{f(\alpha)}{\alpha(\beta-\alpha)}\left\{1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \ldots\ldots + \frac{x^n}{\alpha^n} + \ldots\ldots\right\}$$

$$+ \frac{f(\beta)}{\beta(\alpha-\beta)}\left\{1 + \frac{x}{\beta} + \frac{x^2}{\beta^2} \ldots\ldots + \frac{x^n}{\beta^n} + \ldots\ldots\right\}.$$

In order that the coefficient of x^n may vanish, we must have

$$\frac{f(\alpha)}{\alpha^{n+1}(\beta-\alpha)} + \frac{f(\beta)}{\beta^{n+1}(\alpha-\beta)} = 0, \text{ or } \frac{f(\alpha)}{f(\beta)} = \frac{\alpha^{n+1}}{\beta^{n+1}}$$

therefore $$\log\frac{f(\alpha)}{f(\beta)} + \log\frac{\alpha}{\beta} = n+1.$$

2244. (Proposed by the late W. Lea.)—Form eleven symbols into sets, five symbols in each set, so that every combination of four symbols shall appear once in the sets.

Solution by the Proposer.

Let the symbols be $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}$, and put $n=11$; $p=5$; then the total combinations of four $= \dfrac{n(n-1)(n-2)(n-3)}{2.3.4}$;

and the number contained in each set $= \dfrac{p(p-1)(p-2)(p-3)}{2.3.4}$;

therefore the number of sets $= \dfrac{n(n-1)(n-2)(n-3)}{p(p-1)(p-2)(p-3)} = 66.$

Hence we find that

Each symbol appears $\dfrac{(n-1)(n-2)(n-3)}{(p-1)(p-2)(p-3)} = 30$ times,

Each pair appears $\dfrac{(n-2)(n-3)}{(p-2)(p-3)} = 12$ times,

Each triad appears $\dfrac{n-3}{p-3} = 4$ times.

Proceeding tentatively with the formation of the several sets or combinations of five symbols so as to exhaust in succession the leading symbols a_1, a_2, a_3, a_4, a_5, we get the corresponding numbers of combinations

$$30+18+10+5+3 = 66,$$

the particulars of which are stated in the following table:

30 Combinations, Leading Symbol a_1.

1	$a_1\ a_2\ a_3$	$a_4\ a_5$
2		$a_6\ a_7$
3		$a_8\ a_9$
4		$a_{10}\ a_{11}$
5	$a_1\ a_2\ a_4$	$a_6\ a_9$
6		$a_7\ a_{11}$
7		$a_8\ a_{10}$
8	$a_1\ a_2\ a_5$	$a_6\ a_{10}$
9		$a_7\ a_8$
10		$a_9\ a_{11}$
11	$a_1\ a_2\ a_6$	$a_8\ a_{11}$
12	$a_1\ a_2\ a_7$	$a_9\ a_{10}$
13	$a_1\ a_3\ a_4$	$a_6\ a_8$
14		$a_7\ a_{10}$
15		$a_9\ a_{11}$
16	$a_1\ a_3\ a_5$	$a_6\ a_{11}$
17		$a_7\ a_9$
18		$a_8\ a_{10}$
19	$a_1\ a_3\ a_6$	$a_9\ a_{10}$
20	$a_1\ a_3\ a_7$	$a_8\ a_{11}$
21	$a_1\ a_4\ a_5$	$a_6\ a_7$
22		$a_8\ a_{11}$
23		$a_9\ a_{10}$
24	$a_1\ a_4\ a_6$	$a_{10}\ a_{11}$
25	$a_1\ a_4\ a_7$	$a_8\ a_9$
26	$a_1\ a_5\ a_6$	$a_8\ a_9$
27	$a_1\ a_5\ a_7$	$a_{10}\ a_{11}$
28	$a_1\ a_6\ a_7$	$a_8\ a_{10}$
29		$a_9\ a_{11}$
30	$a_1\ a_8\ a_9$	$a_{10}\ a_{11}$

18 Combinations, Leading Symbol a_2.

31	$a_2\ a_3\ a_4$	$a_6\ a_{11}$
32		$a_7\ a_8$
33		$a_9\ a_{10}$
34	$a_2\ a_3\ a_5$	$a_6\ a_9$
35		$a_7\ a_{10}$
36		$a_8\ a_{11}$
37	$a_2\ a_3\ a_6$	$a_8\ a_{10}$
38	$a_2\ a_3\ a_7$	$a_9\ a_{11}$
39	$a_2\ a_4\ a_5$	$a_6\ a_8$
40		$a_7\ a_9$
41		$a_{10}\ a_{11}$
42	$a_2\ a_4\ a_6$	$a_7\ a_{10}$
43	$a_2\ a_4\ a_8$	$a_9\ a_{11}$
44	$a_2\ a_5\ a_6$	$a_7\ a_{11}$
45	$a_2\ a_5\ a_8$	$a_9\ a_{10}$
46	$a_2\ a_6\ a_7$	$a_8\ a_9$
47	$a_2\ a_6\ a_9$	$a_{10}\ a_{11}$
48	$a_2\ a_7\ a_8$	$a_{10}\ a_{11}$

10 Combinations, Leading Symbol a_3.

49	$a_3\ a_4\ a_5$	$a_6\ a_{10}$
50		$a_7\ a_{11}$
51		$a_8\ a_9$
52	$a_3\ a_4\ a_6$	$a_7\ a_9$
53	$a_3\ a_4\ a_8$	$a_{10}\ a_{11}$
54	$a_3\ a_5\ a_6$	$a_7\ a_8$
55	$a_3\ a_5\ a_9$	$a_{10}\ a_{11}$
56	$a_3\ a_6\ a_7$	$a_{10}\ a_{11}$
57	$a_3\ a_6\ a_8$	$a_9\ a_{11}$
58	$a_3\ a_7\ a_8$	$a_9\ a_{10}$

5 Combinations, Leading Symbol a_4.

59	$a_4\ a_5\ a_8$	$a_9\ a_{11}$
60	$a_4\ a_5\ a_7$	$a_8\ a_{10}$
61	$a_4\ a_6\ a_7$	$a_8\ a_{11}$
62	$a_4\ a_6\ a_8$	$a_9\ a_{10}$
63	$a_4\ a_7\ a_9$	$a_{10}\ a_{11}$

3 Combinations, Leading Symbol a_5.

64	$a_5\ a_6\ a_7$	$a_9\ a_{10}$
65	$a_5\ a_6\ a_8$	$a_{10}\ a_{11}$
66	$a_5\ a_7\ a_8$	$a_9\ a_{11}$

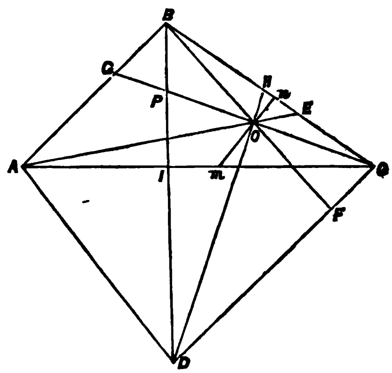
[Note.—After forming the obvious arrangements Nos. 1, 2, 3, 4, put down for subsidiary use the pairs

6	7	8	9
9	11	10	11
10	8	11	10

Then, from an inspection of these, all the arrangements from 5 to 20, both inclusive, are readily found.]

2541. (Proposed by T. SAVAGE, M.A.)—If the diagonals of a convex quadrilateral divide each other in the ratios $\lambda : 1-\lambda$ and $\mu : 1-\mu$, the chance that the line passing through two points taken at random within the quadrilateral will meet both the diagonals at points within the quadrilateral is $\frac{1}{3}\left\{1+2(\lambda+\mu)-2(\lambda^2+\mu^2)\right\}$. Thus, for a parallelogram the chance is $\frac{2}{3}$.

I. *Solution by* STEPHEN WATSON.



Let ABCD be the quadrilateral; I the intersection of the diagonals; and through O, one of the points, draw AE, BF, CG, DH, as in the diagram, and mOn parallel to AB. Then in order that the line through O, P may cut only *one diagonal*, P must lie in one of the triangles AOG, BOH, COE, DOF. Put $AC = a$, $BD = b$, $BI = h$, and $AIB = \alpha$; then when the points O are uniformly distributed over mn, the points G will be uniformly spread over AB, and the same holds good for every other line parallel to mn; hence while O takes every position in the triangle BAC the average distance of G from A is $\frac{1}{2}$AB, and of G from the line AC is $\frac{1}{2}h\sin\alpha$, and the same holds good of the distance of E from AC; also the average distance of O from AC is $\frac{1}{3}h\sin\alpha$. Hence the sum of the areas of the triangles AOG, COE, while O lies in ABC, is

$$a(\tfrac{1}{2}-\tfrac{1}{3})\,h\sin\alpha\,.\,\Delta\,ABC = \tfrac{1}{12}a^2h^2\sin^2\alpha,$$

and the chance of P lying in one of the same triangles is

$$\tfrac{1}{12}a^2h^2\sin^2\alpha \div \tfrac{1}{4}a^2b^2\sin^2\alpha = \frac{h^2}{3b^2} = \tfrac{1}{3}\mu^2;$$

also the like chance when O is in ADC is $= \frac{1}{3}(1-\mu)^2$. Similarly the chance of P lying in one of the triangles BOH, DOF is $\frac{1}{3}\lambda^2$ or $\frac{1}{3}(1-\lambda)^2$ according as O is to the left or right of BD; hence the chance of the line through O, P cutting *one diagonal* is

$$\tfrac{1}{3}\left\{\lambda^2+\mu^2+(1-\lambda)^2+(1-\mu)^2\right\}\ldots\ldots\ldots\ldots\ldots(1);$$

and therefore the chance of cutting *both diagonals* is

$$1-(1)=\tfrac{1}{3}\left\{1+2(\lambda+\mu)-2(\lambda^2+\mu^2)\right\}\ldots\ldots\ldots\ldots(2).$$

In the case of a parallelogram $\lambda=\mu=\frac{1}{2}$, and the chances (1) and (2) become $\frac{1}{3}$ and $\frac{2}{3}$ which are as $1 : 2$.

When the line OP merely *joins* O, P. In this case, in order that OP may cut *both diagonals*, O and P must lie one in the triangle AIB and the other in the triangle CID, the chance of which is $2\lambda\mu(1-\lambda)(1-\mu)$, or one in the

triangle BIC and the other in the triangle AID, the chance of which is the same; hence the chance of cutting *both diagonals* is

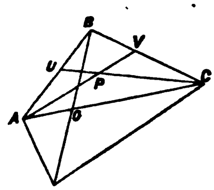
$$4\lambda\mu(1-\lambda)(1-\mu).$$

Similarly the chances of cutting *one diagonal* and *neither diagonal*, are

$$4\lambda\mu(1-\lambda)(1-\mu),\quad 4\left\{\lambda^2\mu^2+(1-\lambda)^2(1-\mu)^2\right\}.$$

In the case of a parallelogram, the three chances just found become $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$, which are as 1 : 1 : 2.

II. *Solution by* CAPT. A. R. CLARKE, R.E., F.R.S.



If through any point P in a triangle ABC, lines APV, CPU be drawn meeting the opposite sides in V and U, it may be easily shown by direct integration, or may be inferred from Professor SYLVESTER's Solution to Question 2371 (*Reprint*, Vol. VIII., p. 36), that (dA being an element of area at P)

$$\int(\text{AUP}+\text{CVP})\,dA = \tfrac{1}{6}(\text{ABC})^2,$$

the integration extending over the area ABC. Hence the total number of pairs of points which determine a line cutting the segment OB of the diagonal BD *only*, is $\frac{1}{6}\cdot\frac{1}{4}\text{OB}^2\cdot\text{AC}^2\cdot\sin^2\text{O}$. Therefore the number of pairs of points determining lines which cut BD *only*, is $\frac{1}{24}(\text{OB}^2+\text{DO}^2)\,\text{AC}^2\sin^2\text{O}$. Consequently the number of pairs of points determining lines which cut only *one* diagonal (within the figure) is

$$\tfrac{1}{24}\left\{(\text{OB}^2+\text{OD}^2)\,\text{AC}^2+(\text{AO}^2+\text{OC}^2)\,\text{BD}^2\right\}\sin^2\text{O}.$$

And the total number of pairs is $\frac{1}{8}\text{AC}^2\cdot\text{BD}^2\sin^2\text{O}$. Hence the chance of cutting one diagonal only is

$$\frac{1}{3}\cdot\frac{\text{OB}^2+\text{OD}^2}{\text{BD}^2}+\frac{1}{3}\cdot\frac{\text{AO}^2+\text{OC}^2}{\text{AC}^2}=\tfrac{1}{3}\left\{\lambda^2+(1-\lambda)^2+\mu^2+(1-\mu)^2\right\}.$$

Subtracting this from unity, we get the stated result for the chance of cutting both diagonals.

2530. (Proposed by Professor CAYLEY.)—Trace the curve $\dfrac{1}{\sqrt{z}}+\dfrac{1}{\sqrt{(x+iy)}}+\dfrac{1}{\sqrt{(x-iy)}}=0$, where the coordinates x, y, z are the perpendicular distances of the current point P from the sides of an equilateral triangle, the coordinates being positive for a point within the triangle.

Solution by the PROPOSER.

The form of the equation shows that the curve is a tricuspidal quartic, having a real cusp at the point ($x=0$, $y=0$), and two imaginary cusps at

the points ($z=0$, $x+iy=0$) and ($z=0$, $x-iy=0$). The rationalised form of the equation is

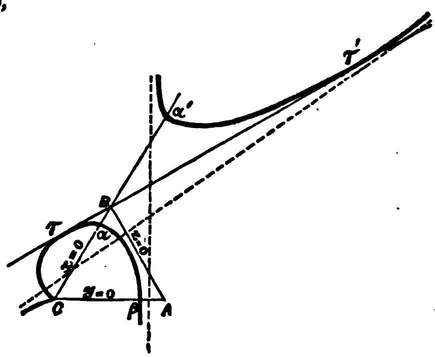
$$(x^2+y^2)^2-4zx\,(x^2+y^2)-4z^2y^2 = 0.$$

$x=0$ gives $y^2(y^2-4z^2)=0$, the point C twice, and two other real points α, α' on the line BC.

$y=0$ gives $x^3(x-4z)=0$, the point C three times, and a real point β on the line CA.

It is easy to find that there is a double tangent $z+2x=0$, viz. $z+2x=0$ gives $(3x^2-y^2)^2=0$, two points τ, τ' (each twice) on the line in question.

Laying down these points, it is easy to see that the curve must have two real asymptotes, and that the form is as shown in the figure.



2524. (Proposed by J. J. WALKER, M.A.)—Show that the equation to the circle circumscribing the triangle formed by the three straight lines $ax+by+c=0$, $a'x\ldots.=0$, $a''x\ldots.=0$ may be thrown into the form

$$(ab'c'')\,(x^2+y^2)+\Sigma\,\frac{(a)^2+(b)^2}{(c)}\,(ax+by+c)=0,$$

where $(ab'c'')$ is the determinant whose constituents are the coefficients in the three linear equations, and $(a)\ (b)\ (c)\ (a')\ldots.$ its first minors.

Solution by the PROPOSER.

1. The condition that four points $(x, y), \ldots., (x_3, y_3)$ should be concyclic (being the analytic expression of Euc. III., 21) is, in rectangular Cartesian coordinates,

$$\frac{(x-x_2)\,(y-y_1)-(x-x_1)\,(y-y_2)}{(x-x_1)\,(x-x_2)+(y-y_1)\,(y-y_2)}-\frac{(x_3-x_2)\,(y_3-y_1)-(x_3-x_1)\,(y_3-y_2)}{(x_3-x_1)\,(x_3-x_2)+(y_3-y_1)\,(y_3-y_2)}=0,$$

or $\quad \Sigma\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}(x^2+y^2)=0$, as is well known.

This may be thrown into the form

$$\begin{aligned}&(1x_2y_3)\,(x^2+y^2)\\ &+\left\{(x_1^2+y_1^2)\,(y_3-y_2)+(x_2^2+y_2^2)\,(y_1-y_3)+(x_3^2+y_3^2)\,(y_2-y_1)\right\}x\\ &+\left\{(x_1^2+y_1^2)\,(x_2-x_3)+(x_2^2+y_2^2)\,(x_3-x_1)+(x_3^2+y_3^2)\,(x_1-x_2)\right\}y\\ &+(x_1^2+y_1^2)\,(x_3y_2-x_2y_3)+(x_2^2+y_2^2)\,(x_1y_3-x_3y_1)+(x_3^2+y_3^2)\,(x_2y_1-x_1y_2)\\ &=0\ldots.(1)\end{aligned}$$

Now if (x_1, y_1) is the intersection of the lines $a'x + \ldots = 0$, $a'x + \ldots = 0$, (x_2, y_2) of $ax + \ldots = 0$, $a''x \ldots = 0$, (x_3, y_3) of $ax + \ldots 0$, $a'x + \ldots = 0$;

then $$x_1 = \frac{(a)}{(c)},\ y_1 = \frac{(b)}{(c)},\ x_2 = \frac{(a')}{(c')},\ y_2 = \frac{(b')}{(c')}\ \ x_3 = \frac{(a'')}{(c'')},\ y_3 = \frac{(b'')}{(c'')},$$

whence $$(1x_2y_3) = \frac{(ab'c'')^2}{(c)\,(c')\,(c'')},\ y_3 - y_2 = \frac{a\,(ab'c'')}{(c')\,(c'')},\ y_1 - y_3 = \frac{a'\,(ab'c'')}{(c)\,(c'')},$$

$$y_2 - y_1 = \frac{a''\,(ab'c'')}{(c)\,(c)'},\ x_2 - x_3 = \frac{b\,(ab'c'')}{(c')\,(c'')},\ x_3 - x_1 = \frac{b'\,(ab'c'')}{(c)\,(c'')},$$

$$x_1 - x_2 = \frac{b''\,(ab'c'')}{(c)\,(c')},\ x_3y_2 - x_2y_3 = \frac{c\,(ab'c'')}{(c)},\ x_1y_3 - x_3y_1 = \frac{c'(ab'c'')}{(c')},$$

$$x_2y_1 - x_1y_2 = \frac{c''\,(ab'c'')}{(c'')}.$$

Substituting these values in the equation (1) above, it becomes divisible by $\frac{(ab'c'')}{(c)\,(c')\,(c'')}$, which reduces it to the form given in the question.

2344. (Proposed by J. Griffiths, M.A.)—Let P be any point on a rectangular hyperbola whose centre is O; then if with centre P and radius PO a circle be described, an infinite number of triangles can be inscribed in the circle, and circumscribed to the hyperbola. Let ABC be any one of these triangles; l, m, n the points where the sides BC, CA, AB touch the hyperbola: it is required to prove that the straight lines Al, Bm, Cn intersect on the circle.

Solution by James Dale.

Taking any one of the triangles ABC as that of reference, the equation to the inscribed conic will be of the form

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The lines joining A, B, C to the points of contact of the opposite sides intersect in the point $lx = my = nz$, which lies on the circumscribing circle if

$$l \sin \mathrm{A} + m \sin \mathrm{B} + n \sin \mathrm{C} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But since (1) is a rectangular hyperbola,

$$l^2 + m^2 + n^2 + 2mn \cos \mathrm{A} + 2nl \cos \mathrm{B} + 2lm \cos \mathrm{C} = 0 \ \ldots\ldots\ldots (3);$$

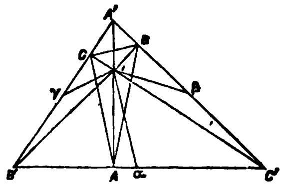
and since (1) passes through the centre of the circumscribing circle, we have

$$l^2 \cos^2 \mathrm{A} + m^2 \cos^2 \mathrm{B} + n^2 \cos^2 \mathrm{C} - 2mn \cos \mathrm{B} \cos \mathrm{C} - \&\mathrm{c.} = 0 \ \ldots\ldots (4).$$

From $\left\{(1) - (2)\right\}^{\frac{1}{2}}$ we obtain $l \sin \mathrm{A} + m \sin \mathrm{B} + n \sin \mathrm{C} = 0$; therefore, by (2), A$l$, B$m$, C$n$ intersect on the circumscribed circle.

2404. (Proposed by J. GRIFFITHS, M.A.)—Let A′, B′, C′ be the centre of the escribed circles of a triangle ABC; I that of the inscribed circle, α, β, γ the middle points of the sides of the triangle A′B′C′. On the segments IA′, IB′, IC′ take X, Y, Z such that $IX = \frac{1}{4}IA'$, $IY = \frac{1}{4}IB'$, $IZ = \frac{1}{4}IC'$, then the circle through X, Y, Z will bisect each of the six segments IA, IB, IC, Iα, Iβ, Iγ.

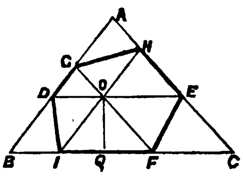
Solution by J. DALE; G. A. OGILVIE; *and others.*



The circle circumscribing ABC is the nine-point circle of A′B′C′, and I is the intersection of the perpendiculars of A′B′C′; therefore the middle points of IA′, IB′, IC′ lie on the circle ABC, and X, Y, Z will lie on a circle of half the dimensions of ABC, having its centre on the middle point of the line joining I to the centre of ABC, and which bisects every line drawn from I to the circumference of ABC (see *Note on the Nine-Point Circle, Reprint*, Vol. VII., p. 86); and which consequently passes through the middle points of IA, IB, IC, Iα, Iβ, Iγ.

2318. (Proposed by S. WATSON.)—Through every point O within a triangle ABC, parallels DE, FG, HI are drawn to BC, CA, AB respectively; show that the average area of the hexagon DIFEHG is three-fourths of the triangle.

Solution by the PROPOSER.



Draw OQ perpendicular to BC; and put DO $= x$, OQ $= y$, and $p =$ the perpendicular from A on BC. Then by similar triangles

$p : a = p - y :$ DE, $\therefore$ DE $= \dfrac{a(p-y)}{p}$, and

$p : a = y :$ IF, $\therefore$ IF $= \dfrac{ay}{p}$; hence the area of the trapezoid IDOF is

$$\tfrac{1}{2}OQ\,(IF + DO) = \frac{y}{2p}(ay + px) \ldots\ldots\ldots\ldots\ldots (1).$$

Now an element at O is $dydx$, and triangle ABC $= \Delta =$ the total number of positions of O; therefore the average area of the trapezoid IDOF is

$$\frac{1}{\Delta}\int_0^p \int_0^{\frac{a(p-y)}{p}} \frac{y}{2p}(ay + px)\,dydx = \frac{\Delta}{4}.$$

Similarly $\frac{1}{4}\Delta$ is the average area of each of the trapezoids EFOH, GHOD; therefore the average area required is $\frac{3}{4}\Delta$.

Otherwise: the area of BDI is $\frac{1}{2}xy$; therefore the average area is

$$\frac{1}{\Delta}\int_0^p \int_0^{\frac{a(p-y)}{p}} \tfrac{1}{2}yx\,dy\,dx = \frac{\Delta}{12},$$

and this is also the average area of each of the triangles CFE, HGA; therefore $\Delta - \frac{1}{4}\Delta$, or $\frac{3}{4}\Delta$, is the average area required, as before.

2522. (Proposed by W. K. Clifford, B.A.)—Prove (1) that the perpendiculars of a circular triangle have a common radical axis; and (2) that if the perpendiculars from the pairs of vertices of one circular triangle on the sides of another meet in a point, then *vice versâ*. (*Def.*—A, B, C being circles, a circle coaxal with A, B, and orthogonal to C, is called the perpendicular from AB on C.)

Solution by the Rev. R. Townsend, F.R.S.

1. If A, B, C be the centres of the three circles forming the triangle; a, b, c their three radii; α, β, γ their three angles of intersection two and two; and X, Y, Z the centres of the three perpendiculars; then since (see *Modern Geometry*, Vol. I., art. 193)

$$\frac{BX}{CX} = \frac{b}{c}\cdot\frac{\cos\gamma}{\cos\beta},\quad \frac{CY}{AY} = \frac{c}{a}\cdot\frac{\cos\alpha}{\cos\gamma},\quad \frac{AZ}{BZ} = \frac{a}{b}\cdot\frac{\cos\beta}{\cos\alpha},$$

therefore at once, by composition of ratios, $\dfrac{BX}{CX}\cdot\dfrac{CY}{AY}\cdot\dfrac{AZ}{BZ} = 1,$

and therefore, &c. [See *Modern Geometry*, Vol. I., art. 190.]

2. If A, B, C be the centres, and a, b, c the radii of the three circles forming one triangle; A′, B′, C′ and a', b', c' those of the three forming the other triangle; X, Y, Z and X′, Y′, Z′ the two triads of centres of the two triads of perpendiculars, and $\widehat{AB'}$ and $\widehat{AC'}$, $\widehat{BC'}$ and $\widehat{BA'}$, $\widehat{CA'}$ and $\widehat{CB'}$ the six angles of intersection of the three pairs of non-corresponding sides of the two triangles; then since (*Mod. Geom.*, Vol. I., art. 193)

$$\frac{BX}{CX} = \frac{b}{c}\cdot\frac{\cos\widehat{BA'}}{\cos\widehat{CA'}},\quad \frac{CY}{AY} = \frac{c}{a}\cdot\frac{\cos\widehat{CB'}}{\cos\widehat{AB'}},\quad \frac{AZ}{BZ} = \frac{a}{b}\cdot\frac{\cos\widehat{AC'}}{\cos\widehat{BC'}},$$

$$\frac{B'X'}{C'X'} = \frac{b'}{c'}\cdot\frac{\cos\widehat{B'A}}{\cos\widehat{C'A}},\quad \frac{C'Y'}{A'Y'} = \frac{c'}{a'}\cdot\frac{\cos\widehat{C'B}}{\cos\widehat{A'B}},\quad \frac{A'Z'}{B'Z'} = \frac{a'}{b'}\cdot\frac{\cos\widehat{A'C}}{\cos\widehat{B'C}},$$

and since consequently, by composition of ratios,

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ} = \frac{B'X'}{C'X'}\cdot\frac{C'Y'}{A'Y'}\cdot\frac{A'Z'}{B'Z'},$$

therefore if either product $=1$, so is the other also, and therefore &c. [*Modern Geometry*, Vol. I., art. 190.)

By inversion from any arbitrary point in space the two preceding properties are at once seen to be true, the former for any arbitrary triad, and the latter for any two arbitrary triads, of circles on the surface of a sphere.

2387. (Proposed by S. ROBERTS, M.A.)—Given a point O and a straight line A, another straight line B passes through O and meets A in a point whose mth polar with regard to a curve of the nth order meets B in $n-m$ points. Required the locus of the points when B revolves about O. Show, by means of the general expression, the locus of the middle points of chords of a conic which pass through a given point.

I. *Solution by* JAMES DALE.

Let (f, g, h) be the given point; (l, m, n) the given line; (x, y, z) a point on the required locus.

The mth polar of any point (x', y', z') on the given line, with regard to a curve of the nth order, is

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)^m\phi=0,$$

the differential symbols operating on ϕ only; and as ϕ is of the order n, this will be of the $(n-m)$th order. Since (x', y', z') lies on the given line, we have the condition $lx'+my'+nz'=0$; and since (f, g, h), (x', y', z'), (x, y, z) lie on the same straight line, we have the condition

$$\begin{vmatrix} y & z \\ g & h \end{vmatrix} x' + \begin{vmatrix} z & x \\ h & f \end{vmatrix} y' + \begin{vmatrix} x & y \\ f & g \end{vmatrix} z' = 0,$$

$$\therefore \frac{x'}{(lf+mg+nh)\,x-f\,(lx+my+nz)}=\frac{y'}{(lf+mg+nh)\,y-g\,(lx+my+nz)}$$
$$=\frac{z'}{(lf+mg+nh)\,z-h\,(lx+my+nz)};$$

and substituting these values in the equation of the polar, we find the equation to the locus of x, y, z to be

$$\left\{(lf+mg+nh)\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\right.$$
$$\left.-(lx+my+nz)\left(f\frac{d}{dx}+g\frac{d}{dy}+h\frac{d}{dz}\right)\right\}^m\phi=0.$$

When ϕ is a conic, this becomes

$$2\phi(x, y, z)=\frac{lx+my+nz}{lf+mg+nh}\left(x\frac{d}{df}+y\frac{d}{dg}+z\frac{d}{dh}\right)\phi,$$

which represents a conic passing through the given point, and the pole of the given line, and through the intersections of the polar of the given point, and the given line with the conic.

When the given line is at infinity the locus becomes

$$\phi(x, y, z) = \frac{ax+by+cz}{4\Delta}\left(x\frac{d}{df}+y\frac{d}{dg}+z\frac{d}{dh}\right)\phi,$$

which represents a conic similar and similarly situated to the given conic, passing through its centre and the given point. This conic is the locus of the middle points of all chords of ϕ which pass through (f, g, h).

II. *Solution by the* PROPOSER.

Using trilinear coordinates (x, y, z), let $z=0$ be the given line A, and let the coordinates of the point O be $(x=0, y=0)$. The line $xy_1-x_1y=0$...(1) passes through O and meets A in the point $(x_1, y_1, 0)$ the mth polar of which relative to the curve of the nth degree $u=0$ is $\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^m u = 0$ ...(2). Eliminate x_1, y_1 between (1) and (2), and we get the required equation

$$\left\{x^m\frac{d^m}{dx^m}+mx^{m-1}\frac{d^m}{dx^{m-1}dy}+\ldots+y^m\frac{d^m}{dy^m}\right\}u = 0 \quad\ldots\ldots\ldots(3).$$

The function here represented is a mixed concomitant of the function u. If we adopt the general forms $\alpha x+\beta y+\gamma z=0$ for the line A and (x_1, y_1, z_1) for the point O, the resulting equation is

$$\left\{\Big((z_1x-x_1z)\gamma-(x_1y-xy_1)\beta\Big)\frac{d}{dx}+\Big((x_1y-xy_1)\alpha-(y_1z-z_1y)\beta\Big)\frac{d}{dy}\right.$$
$$\left.+\Big((y_1z-z_1y)\beta-(z_1x-x_1z)\alpha\Big)\frac{d}{dz}\right\}^m u=0,$$

the differential symbols operating on u only.

In the case of a conic the equation (3) becomes

$$x\frac{du}{dx}+y\frac{du}{dy}=0.$$

And if z is the line at infinity, this represents the locus of the middle points of chords through the origin. (Salmon's *Conics*, p. 95, Ex. 2, 3rd ed.)

2469. (Proposed by R. TUCKER, M.A.) — Transform the equation $x^4+px^3+qx^2+rx+s=0$, with roots $(\alpha, \beta, \gamma, \delta)$, to one whose roots are $\beta\gamma\delta+\beta\gamma+\gamma\delta+\delta\beta+\beta+\gamma+\delta$, &c.

Solution by the PROPOSER; W. CHADWICK; *and others.*

Since $\quad \beta\gamma\delta+\beta\gamma+\gamma\delta+\delta\beta+\beta+\gamma+\delta = \frac{s}{x}+q+x(p+x)-p-x,$

we have to eliminate x between

$$xy = s+(q-p)x+x^2(p-1)+x^3 \text{ and } x^4+px^3+qx^2+rx+s=0;$$

whence we have, if $a = -p+q-r+s$,

$$(a-y)^4+p\,(1+y)\,(a-y)^3+q\,(1+y)^2\,(a-y)^2$$
$$+r\,(1+y)^3\,(a-y)+s\,(1+y)^4 = 0,$$

or
$$(a+1)\,y^4+(r+3p-2q)\,(a+1)\,y^3$$
$$+\left\{a^2\,(q-3p+6)+a\,(3p-4q+3r)+q-3r+6s\right\}y^2$$
$$+\left\{a^3\,(p-4)+a^2\,(2q-3p)+a\,(3r-2q)+4s-r\right\}y$$
$$+a^4+pa^3+qa^2+ra+s = 0.$$

If -1 be a root of the given equation, then the transformed will have three roots equal to -1.

Example: $x^4-4x^3-7x^2+34x-24 = 0$, (roots $2, -3, 4, 1$),
transformed equation $x^4+36x^3-558x^2-30020x-245427 = 0$.

1977. (Proposed by M. W. Crofton, B.A.)—If three lengths are taken at random, find the chance that it will be possible to form a triangle with them; the extreme limit, whatever its magnitude, being supposed the same for all three.

I. *Solution by the* Proposer.

Let AB be the given limit; then the question will be, taking three points 1, 2, 3 at random in AB, to find the chance that A1, A2, A3 may form a triangle.

As the result must be independent of the limit AB, we may take A3, the distance of the farthest point from A, as the limit; *i.e.*, suppose A3 a fixed distance, and 1, 2 taken at random in it. Put $A1 = x$, $A2 = y$, $A3 = a$; then we want the chance that $x+y > a$. Now to every pair of values (x, y) corresponds another pair $(a-x, a-y)$; this we see clearly by conceiving the distances measured from the end 3 of the line A3, instead of from A.

But if $x+y>a$, we have $a-x+a-y<a$.
Hence to every favourable case corresponds one unfavourable case; the chance required is therefore $\frac{1}{2}$.

II. *Solution by* Stephen Watson.

Let x, y, z be the lengths, and a the limiting magnitude. Then, multiplying by 6, the number of ways in which x, y, z can be interchanged, we need only consider the question under the condition $x<y<z$, and thus the possibility of forming a triangle with x, y, z is $x+y>z$; therefore, when $x<\frac{1}{2}z$, the limits of y are $z-x$ and z; but when $x>\frac{1}{2}z$, the limits are x and z. Also the number of lengths taken $=a^3$; hence the required chance is

$$p = \frac{6}{a^3}\int_0^a dz\left\{\int_0^{\frac{1}{2}z} dx\int_{z-x}^{z} dy+\int_{\frac{1}{2}z}^{z} dx\int_x^z dy\right\} = \tfrac{1}{2};$$

therefore the chance is an *even* one whatever be the magnitude of a.

2504. (Proposed by R. TUCKER, M.A.)—O is the centre of the circumscribing circle of a triangle ABC; PK, PL, PM, drawn parallel to CO, BO, AO respectively, meet AB, AC, BC in K, L, M; if KLM be a straight line, then the locus of P will be the conic

$$\frac{a}{\alpha}\cos A\cos(B-C)+\frac{b}{\beta}\cos B\cos(C-A)+\frac{c}{\gamma}\cos C\cos(A-B)=0.$$

I. *Solution by the* Rev. J. L. KITCHIN, M.A.; *the* PROPOSER; *and others.*

The equations to the lines joining O with A, B, C in order, are

$$\beta\cos C-\gamma\cos B=0,\ \ \gamma\cos A-\alpha\cos C=0,\ \ \alpha\cos B-\beta\cos A=0\ \ldots\ldots(1).$$

Three lines through a point $(\alpha_1, \beta_1, \gamma_1)$ parallel to (1) in order, are

$$\left.\begin{array}{l}(\beta\gamma_1-\gamma\beta_1)\cos(B-C)+(\alpha\gamma_1-\gamma\alpha_1)\cos B+(\beta\alpha_1-\alpha\beta_1)\cos C=0\\ (\gamma\alpha_1-\alpha\gamma_1)\cos(C-A)+(\beta\alpha_1-\alpha\beta_1)\cos C+(\gamma\beta_1-\beta\gamma_1)\cos A=0\\ (\alpha\beta_1-\beta\alpha_1)\cos(A-B)+(\gamma\beta_1-\beta\gamma_1)\cos A+(\alpha\gamma_1-\gamma\alpha_1)\cos B=0\end{array}\right\}\ldots\ldots(2).$$

The condition that the intersections of (2) in order with $\alpha=0,\ \beta=0,\ \gamma=0$ may all lie on a straight line is easily found to be

$$\left\{\alpha_1\cos(A-B)+\gamma_1\cos A\right\}\left\{\gamma_1\cos(C-A)+\beta_1\cos C\right\}\left\{\beta_1\cos(B-C)+\alpha_1\cos B\right\}$$
$$+\left\{\beta_1\cos(A-B)+\gamma_1\cos B\right\}\left\{\alpha_1\cos(C-A)+\beta_1\cos A\right\}$$
$$\times\left\{\gamma_1\cos(B-C)+\alpha_1\cos C\right\}=0.$$

Multiplying these out, and performing some obvious reductions, we get

$$\left\{\sin^2 A\cos A\cos(B-C)+\sin^2 B\cos B\cos(C-A)+\sin^2 C\cos C\cos(A-B)\right\}\alpha_1\beta_1\gamma_1$$
$$+\beta_1\gamma_1^2\cos A\cos(B-C)\sin A\sin C+\&c.=0,$$

which, suppressing the accents, splits up into

$$(\alpha\sin A+\beta\sin B+\gamma\sin C)\left\{\beta\gamma\sin A\cos A\cos(B-C)+\&c.\right\}=0.$$

Now, obviously, the first factor put equal to 0 cannot be locus of P; hence the equation of the locus of P is $\beta\gamma\sin A\cos A\cos(B-C)+\&c.=0$, which is the same as that given in the question.

II. *Solution by* STEPHEN WATSON; W. CHADWICK; *and others.*

For brevity let l, m, n stand for $\cos A$, $\cos B$, $\cos C$ respectively, then the equations of AO, BO, CO are contained in

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and those of lines parallel to the same may be written

$$n\beta-m\gamma+p\,(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$l\gamma-n\alpha+p_1(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$
$$m\alpha-l\beta+p_2(a\alpha+b\beta+c\gamma)=0\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation of a line through the intersections of (2) with AC and (3) with AB, is

$$\alpha - \frac{l-bp_2}{m+ap_2}\beta - \frac{l+cp_1}{n-ap_1}\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (5);$$

hence the condition that (2) and (5) shall meet BC in the same point is

$$\frac{m-cp}{n+bp} = -\frac{(l+cp_1)(m+ap_2)}{(n-ap_1)(l-bp_2)},$$

or $2lmn + l(bm-cn)p + m(cn-al)p_1 + n(al-bm)p_2$

$$+c(al+bm)pp_1 + a(bm+cn)p_1p_2 + q(cn+al)p_2p = 0 \;\ldots\ldots (6);$$

therefore, putting in this the values of p, p_1, p_2 derived from (2), (3), (4), the coefficients of $\alpha^2, \beta^2, \gamma^2$ will be found to vanish in the result; the other coefficients have the common factor $al+bm+cn$ which may therefore be omitted; and then the coefficient of $\beta\gamma$ is

$$l(bm+cn) = \tfrac{1}{2}a\cot A(\sin 2B + \sin 2C) = a\cos A\cos(B-C),$$

with similar expressions for the coefficients of $\gamma\alpha$ and $\alpha\beta$; hence the equation of the required locus is

$$\frac{a}{\alpha}\cos A\cos(B-C) + \frac{b}{\beta}\cos B\cos(C-A) + \frac{c}{\gamma}\cos C\cos(A-B) = 0.$$

2479. (Proposed by R. Ball, M.A.)—Determine m so that

$$\frac{-3b+am}{2a} + \frac{\sqrt{(9b^2-12ac-6abm-3a^2m^2)}}{2a}$$

shall be a root of the cubic $ax^3+3bx^2+3cx+d=0$.

Solution by Samuel Roberts, M.A.

A root of the cubic is of the form $-\dfrac{b+N}{a}$, where N is a known function of the coefficients. Comparing this with the given form, we have

$$\frac{m}{2} + \frac{b}{2a} + \frac{\sqrt{\phi}}{2a} = \frac{N}{a}$$

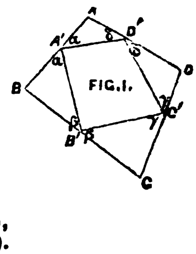
and therefore $\qquad \phi = (am+b-2N)^2,$

or $\qquad 4a^2m^2 + (8b-4N)m + (b-2N)^2 - 9b^2 + 12ae,$

whence $\qquad 2am = -(2b-N) + \sqrt{\left\{12(b^2-ae) - 3N^2\right\}}.$

2288. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—In Quest. 1705 *Reprint*, Vol. III., p. 109), it is stated, that "if from the intersection of the diagonals of a quadrilateral inscribed in a circle perpendiculars be drawn on the sides, the quadrilateral formed by joining the feet of these perpendiculars is of all quadrilaterals inscribed in the given one, the one of least perimeter." Discuss the more general case in which the given quadrilateral is not inscribable in a circle.

Solution by the PROPOSER.



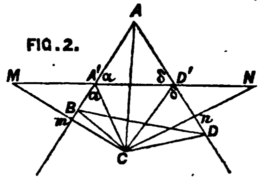
It is somewhat remarkable that the inscription of a quadrilateral of minimum perimeter within a given quadrilateral is possible only when the given quadrilateral is inscribable in a circle. When the perimeter of an inscribed quadrilateral A'B'C'D' is a minimum, its sides must make equal angles with each side of the given quadrilateral ABCD. Let these equal angles be denoted by $\alpha, \beta, \gamma, \delta$ as marked in the diagram (Fig. 1) and we shall have these relations amongst the several angles, viz.,

$$A = \pi-(\alpha+\delta), \quad B = \pi-(\alpha+\beta),$$
$$C = \pi-(\beta+\gamma), \quad D = \pi-(\gamma+\delta).$$

Therefore $\qquad A+C = B+D$; but $A+C+B+D = 2\pi$;

whence $A+C = B+D = \pi$, and the quadrilateral is inscribable in a circle. What therefore becomes of the general case in which ABCD is not so inscribable? On examination it will be found that one side B'C' of the inscribed quadrilateral vanishes, forming a double point at the angle C, and that the figure thus merges into a triangle CA'D'. Draw Cm (Fig. 2) perpendicular to AB and produce it to M, making Cm = mM; similarly draw CnN perpendicular to AD, making Cn = nN. Then M and N are reflected images of the angular point C with respect to the sides AB, AD. Join MN cutting these sides in A', D'; and join CA', CD'. Then CA', A'D' make equal angles (α) with AB, and CD', A'D' make equal angles (δ) with AD. The direction of the zero side which connects the double point at C is indeterminate, and may therefore satisfy any condition, but it is not competent to satisfy two conditions that are independent. The quadrilateral A'$\overline{\text{CC}}$D', though not strictly of minimum perimeter according to analytical theory, is one of least perimeter, because the value of this perimeter would become increased by a small displacement of any one of the four points.

FIG. 2.



The perimeter $= CA'+A'D'+D'C = MA'+A'D'+D'N = MN$, which is the distance between the reflected images of C with respect to AB, AD, and is independent of the positions of the points B, D on those lines.

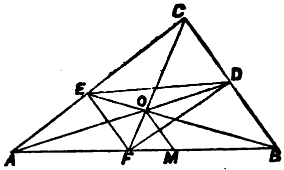
Again, since the angle $A'CD' = \pi - CA'D' - A'D'C = \pi-(\pi-2\alpha)-(\pi-2\delta)$ $= 2(\alpha+\delta)-\pi = \pi-2A$, it follows that the construction will require the angle A to be acute; also that each acute angle of the given quadrilateral will determine a separate construction similar to that we have indicated.

COR.—When the given quadrilateral is inscribable in a circle in accordance with Quest. 1705, there are innumerable inscribed quadrilaterals of minimum perimeter, having their sides parallel and the perimeters equal to one another.

The construction we have given with a double point will be one of these, and will assist in determining the value of this perimeter, which according to what precedes is the distance of the reflected images of any angle with respect to the opposite sides. The distance of the reflected images is therefore of the same magnitude for each angle. For in this case we have the angle $MCN = BCD$; also $CM = 2BC \sin B$, and $CN = 2CD \sin D = 2CD \sin B$; therefore $MN = 2BD \sin B$, and if R denote the radius of the circumscribing circle, $R \cdot MN = AC \cdot BD$. That is, the rectangle under the radius and the minimum perimeter is equal to the rectangle under the diagonals of the given quadrilateral.

2537. (Proposed by the EDITOR.)—A point O is taken within a triangle ABC, and straight lines AOD, BOE, COF, are drawn through O, meeting the sides in D, E, F; show that the average area of the triangle DEF, in parts of the area of the triangle ABC, is $10-\pi^2$.

I. *Solution by* CAPTAIN A. R. CLARKE, R.E., F.R.S.



First to determine the average area of FDB. The following relations are obvious amongst the areas of the triangles,

$$\frac{\text{FDB}}{\text{DOB}}=\frac{\text{FC}}{\text{OC}},\quad \frac{\text{DOB}}{\text{AOB}}=\frac{\text{DO}}{\text{OA}},\quad \frac{\text{AOB}}{\text{ACB}}=\frac{\text{OF}}{\text{FC}}.$$

Multiplying these together we find that the area of FDB, expressed in parts of the area of triangle ABC ($=\Delta$), is

$$\text{FDB}=\frac{\text{FO}\,.\,\text{OD}}{\text{CO}\,.\,\text{OA}}\dots\dots\dots\dots\dots\dots\dots\dots\dots(\alpha).$$

From O draw OM parallel to BC, meeting AB in M, and let AM $=x$ and MO $=y$ be the coordinates of O; then (α) becomes

$$\text{FDB}=\frac{y}{a-y}\,.\,\frac{c-x}{x}.$$

Consequently the average area of FDB is

$$\frac{1}{\Delta}\int_0^a\int_{\frac{cy}{a}}^{c}\frac{y}{a-y}\left(\frac{c}{x}-1\right)\sin\text{B}\,.\,dy\,dx$$

$$=2\int_0^1\frac{\frac{y}{a}}{1-\frac{y}{a}}\left(-\log\frac{y}{a}-1+\frac{y}{a}\right)d\left(\frac{y}{a}\right)\dots\dots(\beta).$$

Now, by expanding $(1-\phi)^{-1}$ in series, it is easy to show that

$$\int_0^1\frac{\phi}{1-\phi}\log\phi\,.\,d\phi=1-\frac{\pi^2}{6};$$

hence (β) becomes $=2\left(\frac{1}{6}\pi^2-1-\frac{1}{2}\right)$, and consequently the average area of the triangle DEF, in terms of the area of the original triangle ABC, is three times this quantity subtracted from unity, that is $10-\pi^2$, or ·1304 nearly.

[In the Editorial *Note* to the solution of Question 1683 (*Reprint*, Vol. V., p. 31) it is shown that the area of DEF, expressed in triangular coordinates (x, y, z), is $\frac{2xyz}{(1-x)(1-y)(1-z)}$; consequently the average area required is

$$2\int_0^1\int_0^{1-x}dx\,dy\,\frac{2xy(1-x-y)}{(1-x)(1-y)(x+y)}=4\int_0^1 dx\left(x+\frac{x^2\log x^2}{1-x^2}\right)\dots(\gamma),$$

$$(\text{putting } x=\epsilon^{-z})=2-8\int_0^\infty dz\,.\,z\left(\epsilon^{-3z}+\epsilon^{-5z}+\epsilon^{-7z}+\dots\dots\right)\dots\dots(\delta),$$

$$=2-8\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\dots\right)=2-8\left(\frac{\pi^2}{8}-1\right)=10-\pi^2.$$

If the points D, E are taken arbitrarily, instead of the point O, Captain

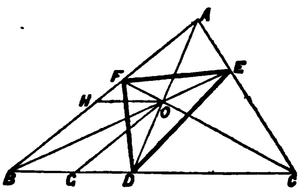
CLARKE has found (see Quest. 2576, p. 52 of this volume) that the average area of the triangle DEF is $\frac{1}{16}(\pi^2-8)$, or ·1168 nearly.

If we proceed to the analogous problem in space of three dimensions, (viz., A point O is taken arbitrarily within a tetrahedron ABCD, and straight lines AOE, BOF, COG, DOH are drawn, meeting the opposite faces in E, F, G, H; to find the average volume of the tetrahedron EFGH,) we shall find, by the same method as that used in the *Note* above referred to, that the volume of EFGH, expressed in tetrahedral coordinates (w, x, y, z) is $\frac{3wxyz}{(1-w)(1-x)(1-y)(1-z)}$; hence the mean volume of EFGH, in parts of the volume of ABCD, will be given by the expression

$$6\int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz\,\frac{3xyz(1-x-y-z)}{(1-x)(1-y)(1-z)(x+y+z)}\ldots(\epsilon),$$

which does not seem to admit of integration in finite terms.]

II. *Solution by* STEPHEN WATSON.



Draw OG, OH respectively parallel to AB, BC, and put CG $= x$, AH $= y$, and triangle ABC $= \Delta$.

Then $\qquad x : c-y = a : \text{BF},$

therefore $\qquad \text{BF} = \frac{a(c-y)}{x}.$

Similarly $\qquad \text{BD} = \frac{c(a-x)}{y}.$

Now an element of the triangle at O is $\sin B\, dx\, dy$; moreover the limits are y from $\frac{c(a-x)}{a}$ to c, and of x from 0 to a.

Hence the average area of the triangle BDF is

$$\frac{\sin^2 B\iint \text{BD}\,.\,\text{BF}\,.\,dx\,dy}{2\Delta} = \sin B\int_0^a dx\int_{\frac{c(a-x)}{a}}^{c}\frac{(a-x)(c-y)}{xy}\,dy$$

$$= \sin B\int_0^a\left(-\frac{cx}{a} - c\log\frac{a-x}{a}\right)\frac{a-x}{x}\,dx$$

$$= -\Delta + c\sin B\int_0^a \log\frac{a-x}{a}\,dx - 2\Delta\int_0^a\log\left(1-\frac{x}{a}\right)\frac{dx}{x}$$

$$= -3\Delta + 2\Delta\int_0^a\left\{\frac{x}{a} + \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \&c.\right\}\frac{dx}{x}$$

$$= -3\Delta + 2\Delta\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c.\right)$$

$$= -3\Delta + \frac{\pi^2}{3}\Delta.$$

The result will be the same for the average area of each of the triangles CDE, AEF; hence the average area of the triangle DEF is

$$\Delta - 3\left(-3\Delta + \frac{\pi^2}{3}\Delta\right) = (10-\pi^2)\Delta.$$

III. *Solution by* SAMUEL BILLS.

1. For the plane problem enunciated in the question.

Referring to the diagram of the foregoing Solution, and putting BG$=x$, GO$=y$, the area of the triangle DEF, in parts of the area of ABC, is

$$\frac{\text{area DEF}}{\text{area ABC}}=\frac{2xy\,(ac-ay-cx)}{(a-x)\,(c-y)\,(ay+cx)},\ (\textit{Reprint}, \text{Vol. V., p. 29}) \ldots\text{(A)}.$$

Now an element at O is $\sin B\,dx\,dy$; also the measure of the number of positions of O, that is to say, of the total number of inscribed triangles, is $\Delta ABC = \frac{1}{2}ac\sin B$; hence the average required is

$$\frac{4}{ac}\int_0^a\int_0^{\frac{c}{a}(a-x)} dx\,dy\,\frac{xy\,(ac-ay-cx)}{(a-x)\,(c-y)\,(ay+cx)}\ \ldots\ldots\ldots\ldots\text{(B)}.$$

Integrating with respect to y, (B) becomes

$$\frac{4}{a}\int_0^a dx\left(\frac{x}{a}+\frac{2x^2}{a^2-x^2}\log\frac{x}{a}\right)\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

[Mr. BILLS integrates (C) by a rather long process; if, however, we put $x=a\epsilon^{-z}$, the expression (C) will reduce to the form marked (δ) in the Editorial *Note* to Capt. CLARKE's Solution.]

2. For the analogous problem in regard to a tetrahedron, enunciated in the Editorial *Note* at the end of the first Solution.

Take the vertex A of the tetrahedron as origin of Cartesian coordinates, and the edges AB ($=b$), AC ($=c$), AD ($=d$) as axes of x, y, z respectively; and put (x, y, z) for the coordinates of O, (α, β, γ) for the plane angles meeting at the solid angle O, and

$$m = (1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}};$$

then, finding the coordinates of E, F, G, H in the usual way, and substituting their values in the formula for the volume of a tetrahedron in terms of the coordinates of its vertices, we find that the volume of the inscribed tetrahedron EFGH, expressed in parts of the volume of ABCD ($=\frac{1}{6}mbcd$) is

$$\frac{\text{vol. EFGH}}{\text{vol. ABCD}}=\frac{3xyz\,(bcd-cdx-dby-bcz)}{(b-x)\,(c-y)\,(d-z)\,(cdx+dby+bcz)}\ \ldots\ldots\ldots\text{(D)}.$$

Now an element at O is $m\,dx\,dy\,dz$; also the measure of the number of positions of O is vol. ABCD $=\frac{1}{6}mbcd$; hence the average volume of the tetrahedron EFGH, in parts of the volume of ABCD, will be given by the expression

$$\frac{6}{bcd}\int_0^b\int_0^{c\left(1-\frac{x}{b}\right)}\int_0^{d\left(1-\frac{x}{b}-\frac{y}{c}\right)} dx\,dy\,dz\,\frac{3xyz\,(bcd-cdx-dby-bcz)}{(b-x)(c-y)\,(d-z)\,(cdx+dby+bcz)}$$

$$\ldots\ldots\ldots\ldots\text{(E)}.$$

I have tried this integral by a variety of methods, but have not been able to find its value in any way.

[If in Mr. BILLS's investigation we suppose the triangle ABC to have the sides BA and BC each equal to unity, as we may do without loss of generality, the expressions (B) and (C) become (by making $a=c=1$) identical with those

marked (γ) in the *Note* at the end of the first Solution. Similarly in the tetrahedron, if we suppose the edges AB, AC, AD to be each equal to unity, the expression (E) becomes (by making in it $b=c=d=1$) identical with that marked (ϵ) in the *Note* to the first Solution.]

2576. (Proposed by Captain CLARKE, F.R.S.)—If in Question 2537 the points D, E be taken arbitrarily, instead of the point O, show that the average area of the triangle DEF, in parts of the area of ABC, is $\frac{1}{16}(\pi^2-8)$.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.

Referring to the diagram in the solution to Question 2537 (see p. 49 of this volume), let $\frac{BD}{BC}=x$, $\frac{CE}{CA}=y$, $\frac{AF}{AB}=z$; then $xyz=(1-x)(1-y)(1-z)$, therefore the area of the triangle DEF in parts of the area of ABC, is

$$\triangle DEF = 1-x(1-z)-y(1-x)-z(1-y)=\frac{2xy(1-x)(1-y)}{1-x-y+2xy}\ldots\ldots(\alpha).$$

Now x and y being equally likely to have any value from 0 to 1, the average value of (α) is $\int_0^1\int_0^1(\alpha)\,dx\,dy$, which, if we put $2x=1+x'$ and $2y=1+y'$, becomes, suppressing the unnecessary accents,

$$\frac{1}{16}\int_{-1}^{+1}\int_{-1}^{+1}\frac{(1-x^2)(1-y^2)}{1+xy}\,dx\,dy$$

$$=\frac{1}{8}\int_{-1}^{1}\frac{1-x^2}{x^2}\left(1-\frac{1-x^2}{2x}\log\frac{1+x}{1-x}\right)dx\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

$$=\frac{1}{8}\int_{-1}^{1}\frac{1-x^2}{x^3}\left\{x-(1-x^2)\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)\right\}dx$$

$$=\frac{1}{8}\int_{-1}^{1}\left\{(1-\tfrac{1}{3})-x^2(1-\tfrac{2}{3}+\tfrac{1}{5})-x^4(\tfrac{1}{3}-\tfrac{2}{5}+\tfrac{1}{7})-\ldots\right\}dx$$

$$=\frac{1}{4}\left\{(1-\tfrac{1}{3})-\tfrac{1}{3}(1-\tfrac{2}{3}+\tfrac{1}{5})-\tfrac{1}{5}(\tfrac{1}{3}-\tfrac{2}{5}+\tfrac{1}{7})-\ldots\right\}$$

$$=\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)+\frac{1}{4}\left(1-\frac{2}{1.3}-\frac{2}{3.5}-\frac{2}{5.7}-\ldots=0\right)=\frac{\pi^2-8}{16}.$$

[Capt. CLARKE and Mr. WATSON integrate the expression (β) in the following manner. Since this function of x remains unchanged when $-x$ is

written for x, it will be sufficient to integrate from $x=0$ to $x=1$, and double the result; thus (β) becomes

$$\frac{1}{4}\int_0^1\left(\frac{1-x^2}{x^2}-\frac{(1-x^2)^2}{2x^3}\log\frac{1+x}{1-x}\right)dx=\frac{1}{8}\left[\frac{1-x^4}{2x^2}\log\frac{1+x}{1-x}-\frac{1}{x}-3x\right]_0^1$$
$$+\frac{1}{4}\int_0^1\frac{1}{x}\log\frac{1+x}{1-x}\,dx.$$

Now $$\frac{1-x^4}{2x^3}\log\frac{1+x}{1-x}-\frac{1}{x}=\frac{x}{3}-4\left(\frac{x^3}{1.5}+\frac{x^5}{3.7}+\frac{x^7}{5.9}+\dots\right),$$
which is -1 when x is unity, and 0 when x is zero. Also

$$\int_0^1\frac{1}{x}\log\frac{1+x}{1-x}\,dx=2\int_0^1\left(1+\frac{x^2}{3}+\frac{x^5}{5}+\dots\right)dx=\frac{\pi^2}{4};$$

hence the average required is $\frac{1}{16}(\pi^2-8)$.]

2575. (Proposed by C. W. MERRIFIELD, F.R.S.)—Given two surfaces of the second degree, there exists *in general** a set of Cartesian axes, whose directions are those of conjugate diameters in every one of the surfaces of the second degree passing through the intersection* of the two surfaces given. The directions will be those of the three paraboloids* of the system of surfaces. The locus of centres of the system lies on the intersection of three cylinders of the second degree whose axes are parallel to those of the paraboloids of the system.

I. *Solution by* Professor TAIT.

A solution of this question is readily obtained by *Quaternions;* at least as regards *central* surfaces, and it is quite easy to extend it to the others. For any surface through the intersection of

$S\rho\phi\rho=1$ and $S(\rho-\alpha)\psi(\rho-\alpha)=e$, is $kS\rho\phi\rho-S(\rho-\alpha)\psi(\rho-\alpha)=k-e$,

where k and e are scalars.

The axes of this depend only on the term $S\rho(k\phi-\psi)\rho$.
Hence the set of conjugate diameters which are the same in all are the roots of

$$V(k\phi-\psi)\rho(k_1\phi-\psi)\rho=0,\ \text{or}\ V\phi\rho\psi\rho=0,$$

as we might have seen without analysis.

The locus of the centres is given by the equation

$$(\psi-k\phi)\rho-\psi\alpha=0,$$

where k is a scalar variable.

These equations give the required results at once.

* Real or imaginary: the reality of the reductions is another matter.

II. *Solution by the* REV. JOSEPH WOLSTENHOLME, M.A.

Let the equation of one surface be

$$ax^2+by^2+cz^2=1 \dots\dots\dots\dots\dots\dots\dots\dots (1);$$

and let one be described similar, and similarly situated to the second, but concentric with this; and let the equation of this be

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=1 \dots\dots\dots (2).$$

Now, if a plane $lx+my+nz=0$ be taken, the conjugate diameters with respect to (1) and (2) are

$$\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}; \quad \text{and} \quad \frac{Ax+Hy+Gz}{l}=\frac{Hx+By+Fz}{m}=\frac{Gx+Fy+Cz}{m}.$$

If these coincide, and (λ, μ, ν) be the direction, we have

$$A\lambda+H\mu+G\nu=ka\lambda, \ \&c.,$$

giving the cubic $\begin{vmatrix} A-ka, & H, & G \\ H, & B-kb, & F \\ G, & F, & C-kc \end{vmatrix}=0$; which, when developed, is

$$(A-ka)(B-kb)(C-kc)-F^2(A-ka)-G^2(B-kb)-H^2(C-kc)+2FGH=0.$$

This may be written in the form

$$(k-u)(k-v)(k-w)-\frac{GH}{aF}(k-v)(k-w)-\dots\dots=0;$$

where $\quad A-\dfrac{GH}{F}=au, \quad B-\dfrac{HF}{G}=bv, \quad C-\dfrac{FG}{H}=cw,$

and it can be readily shown to have three real roots, if abcFGH be positive; but this is not a criterion.

Supposing k_1, k_2, k_3 to be the roots, and $(\lambda_1, \mu_1, \nu_1)$, &c. the corresponding values of (λ, μ, ν); then $A\lambda_1+H\mu_1+G\nu_1=k_1a\lambda_1$, &c.

Multiplying by λ_2, μ_2, ν_2, and adding, we get

$$\lambda_1(A\lambda_2+H\mu_2+G\nu_2)+\dots=k_1(a\lambda_1\lambda_2+b\mu_1\mu_2+c\nu_1\nu_2), \text{ or } (k_1-k_2)(a\lambda_1\lambda_2+\dots)=0;$$

or except when the above cubic has equal roots, the three directions corresponding to the three roots form a system of conjugate diameters for (1), and similarly for (2). We get therefore for any two conicoids one system of axes parallel to a system of conjugate diameters for both, and referred to these the equations of the two will be

$$u \equiv \alpha x^2+\beta y^2+\gamma z^2-1=0, \quad v \equiv \alpha' x^2+\dots+2\alpha'' x+\dots=0;$$

and the general equation of a conicoid through their intersections is $u+rv=0$; hence any such conicoid has a system of conjugate diameters parallel to the axes of coordinates. Also, if we eliminate successively x^2, y^2, z^2, we shall get the equations of three paraboloids through the curve of intersection, and their axes are parallel to the axes of coordinates.

The locus of centres is $\dfrac{du}{dx}+\dfrac{dv}{dx}=\dfrac{du}{dy}+\dfrac{dv}{dy}=\dfrac{du}{dz}+\dfrac{dv}{dx}$; and any one of these equations gives a hyperbolic cylinder with its axis parallel to one of the coordinate axes.

III. *Solution by the* PROPOSER.

The two surfaces may be written, without any loss of generality upon their intersection, as

$$U=a_1x^2+b\,.\,y^2+c_1z^2+1=0.$$
$$V=a_2(x-\alpha)^2+b_2(y-\beta)^2+c_2(z-\gamma)^2+1=0.$$

For these contain nine independent constants, and there are nine more involved in the choice of axes;—viz., 3 for origin, 3 for inclination of axes, and 3 for aspect, making 18 in all, which is just the number in two quaternary quadrics of the most general form.

Now, $a_2U - a_1V = 0$, $b_2U - b_1V = 0$, $c_2U - c_1V = 0$ are three paraboloids of which the axes are parallel to those of x, y, and z. The form of the equations shows that the axes are parallel to conjugate diameters; and, k being indeterminate, $U + kV = 0$ is of the same form as V, becoming parabolic when
$$k = -\frac{a_1}{a_2},\ -\frac{b_1}{b_2},\ \text{or}\ -\frac{c_1}{c_2}.$$

Again, the coordinates (x, y, z) of the centre of $U + kV = 0$ are
$$x = \frac{a_2 \alpha k}{a_1 + ka_2},\quad y = \frac{b_2 \beta k}{b_1 + kb_2},\quad z = \frac{c_2 \gamma k}{c_1 + kc_2};$$
therefore
$$k = \frac{a_1 x}{a_2(\alpha - x)} = \frac{b_1 y}{b_2(\beta - y)} = \frac{c_1 z}{c_2(\gamma - z)},$$
which is the intersection of three cylinders of the second degree, written in hyperbolic form, whose axes are parallel to those of reference.

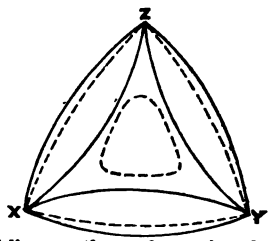
2553. (Proposed by Professor CAYLEY.)—Show that the surface $y^2z^2 + z^2x^2 + x^2y^2 - 2xyz = 0$ meets the sphere $x^2 + y^2 + z^2 = 1$ in four circles; and explain in a general manner the form of the curve of intersection of the surface by any other sphere having the same centre, and thence the form of the surface itself (being a particular case of Steiner's surface, and which by the homographic transformations $w^{-1}x$, $w^{-1}y$, $w^{-1}z$, for x, y, z gives $y^2z^2 + z^2x^2 + x^2y^2 - 2wxyz = 0$, the general equation of Steiner's surface).

Solution by the PROPOSER.

Take X, X′, Y, Y′, Z, Z′ the intersections of the sphere $x^2 + y^2 + z^2 = 1$ by the three axes respectively; then we have $x^2 + y^2 + z^2 = 1$, $x + y + z = -1$, the equations of the circle through the points X′, Y′, Z′; and from these two equations we deduce $yz + zx + xy = 0$, and thence
$$y^2z^2 + z^2x^2 + x^2y^2 + 2xyz\,(x + y + z) = 0,$$
that is
$$y^2z^2 + z^2x^2 + x^2y^2 - 2xyz = 0;$$
so that the circle lies on the quartic surface; and by changing successively the signs of each two of the three coordinates, we have three other circles lying on the sphere and also on the quartic surface; viz., we have in all four circles, the above mentioned circle through (X′, Y′, Z′), and three other circles through (X′, Y, Z), (X, Y′, Z), (X, Y, Z′) respectively, making together a curve of the order 8, the complete intersection of the quartic surface by the sphere.

The quartic surface lies entirely in the four octants of space xyz, $xy'z'$, $x'yz'$, $x'y'z$; and as to the portion of the surface which lies in the octant xyz, this meets the sphere $x^2 + y^2 + z^2 = 1$ in portions of the three circles (X′, Y, Z) (X, Y′, Z) (X, Y, Z′) constituting a tricuspidal form lying within the octant XYZ as shown in the figure. The intersection by a sphere, radius

<1, projected on the octant XYZ, is a trinodal form, lying outside the tricuspidal one, as shown by a dotted line in the figure; the intersection by a sphere radius >1, projected in the same way, is a trigonoid form lying inside the tricuspidal one, as also shown by a dotted line in the figure; as the radius approaches to and ultimately becomes $=\frac{2}{\sqrt{3}}$, this diminishes, and becomes ultimately a mere point, and when the radius is greater than this value the intersection is imaginary.



Imagine on the solid sphere, radius $=1$, the four tricuspidal forms lying in alternate octants as above; cut away down to the centre the portions lying without these tricuspidal forms; and build up on the tricuspidal forms, until the greatest distance from the centre becomes $=\frac{2}{\sqrt{3}}$; we have a solid figure with four prominences situate as the summits of a tetrahedron, the bounding surface whereof is the surface in question: it is to be added that the axes are nodal lines on the surface, viz., the portions which lie within the solid figure are the intersections of two real sheets of the surface, the portions which lie without the solid figure are isolated, or acnodal, lines on the surface.

2481. (Proposed by the Rev. J. Blissard.)—Find the developments of $\frac{(\sin^{-1}x)^{2n-1}}{\Gamma(2n)}$ and $\frac{(\sin^{-1}x)^{2n}}{\Gamma(2n+1)}$, and show that they possess a property analogous to that noticed by Professor Sylvester (see Question 2447)* as belonging to the developments of $(\sin^{-1}x)$ and $\frac{1}{2}(\sin^{-1}x)^2$.

Solution by the Proposer.

The required developments may be arrived at either by an inductive or a deductive method. The former being the simpler, we shall use it here, and proceed step by step.

1. Let $\quad x = P_1 \sin x + P_3 \frac{\sin^3 x}{1.2.3} + \ldots + P_{2n-1} \frac{\sin^{2n-1} x}{1.2\ldots(2n-1)} + \ldots.$

Differentiate twice, then we have

$$0 = -\left(P_1 \sin x + P_3 \frac{\sin^3 x}{1.2} + \ldots\right) + (1-\sin^2 x)\left(P_3 \sin x + P_5 \frac{\sin^3 x}{1.2.3} + \ldots\right);$$

therefore $\quad \frac{P_{2n+1}}{1.2\ldots(2n-1)} - P_{2n-1}\left(\frac{1}{1.2\ldots(2n-2)} + \frac{1}{1.2\ldots(2n-3)}\right) = 0;$

* *Reprint*, Vol. VIII., p. 59.

therefore $P_{2n+1} = (2n-1)^2 P_{2n-1} = (2n-1)^2(2n-3)^2 P_{2n-3} \ldots$
$= (2n-1)^2(2n-3)^2 \ldots 3^2 . 1^2 . P_1;$

but $P_1 = 1$, hence we have

$$\frac{P_{2n+1}}{1.2\ldots(2n+1)} = \frac{1^2.3^2\ldots(2n-1)^2}{1.2\ldots(2n+1)} = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \cdot \frac{1}{2n+1}.$$

Hence $$x = \sin x + \frac{1}{2} \cdot \frac{\sin^3 x}{3} + \frac{1.3}{2.4} \cdot \frac{\sin^5 x}{5} + \ldots;$$

and putting $\sin^{-1} x$ for x, $\sin^{-1} x = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$

2. Let $$\frac{x^2}{1.2} = P_2 \frac{\sin^2 x}{1.2} + P_4 \frac{\sin^4 x}{1.2.3.4} + \ldots.$$

Differentiating twice, we have

$$1 = -\sin x \left(P_2 \frac{\sin x}{1} + P_4 \frac{\sin^3 x}{1.2.3} + \ldots\right) + (1-\sin^2 x)\left(P_2 + P_4 \frac{\sin^2 x}{1.2} + \ldots\right).$$

Hence $P_{2n+2} = (2n)^2 P_{2n} = (2n)^2(2n-2)^2 \ldots 4^2 . 2^2 . P_2$; but $P_2 = 1$,

therefore $$\frac{P_{2n+2}}{1.2\ldots(2n+2)} = \frac{2^2.4^2\ldots(2n)^2}{1.2\ldots(2n+2)} = \frac{2.4\ldots(2n)}{3.5\ldots(2n+1)} \cdot \frac{1}{2n+2},$$

therefore $$\frac{x^2}{1.2} = \frac{\sin^2 x}{2} + \frac{2}{3} \cdot \frac{\sin^4 x}{4} + \frac{2.4}{3.5} \cdot \frac{\sin^6 x}{6} + \ldots,$$

and $$\frac{(\sin^{-1} x)^2}{1.2} = \frac{x^2}{2} + \frac{2}{3} \cdot \frac{x^4}{4} + \frac{2.4}{3.5} \cdot \frac{x^6}{6} + \ldots.$$

3. Let $$\frac{x^3}{1.2.3} = P_3 \frac{\sin^3 x}{1.2.3} + P_5 \frac{\sin^5 x}{1.2\ldots5} + \ldots.$$

Differentiating twice, and substituting $\sin^{-1} x$ for x, we have

$$\sin^{-1} x = -x\left(P_3 \frac{x^2}{1.2} + P_5 \frac{x^4}{1.2.3.4} + \ldots\right) + (1-x^2)\left(P_3 x + P_5 \frac{x^3}{1.2.3} + \ldots\right);$$

but $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \&c.$; therefore, equating coefficients of x^{2n+1},

$$\frac{P_{2n+3}}{1.2\ldots(2n+1)} - P_{2n+1}\left(\frac{1}{1.2\ldots 2n} + \frac{1}{1.2\ldots(2n-1)}\right) = \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \cdot \frac{1}{2n+1},$$

therefore $$P_{2n+3} = (2n+1)^2 P_{2n+1} + 1^2.3^2\ldots(2n+1)^2 \cdot \frac{1}{(2n+1)^2},$$

and $$P_{2n+1} = (2n-1)^2 P_{2n-1} + 1^2.3^2\ldots(2n-1)^2 \cdot \frac{1}{(2n-1)^2},$$

$$\therefore P_{2n+3} = (2n+1)^2(2n-1)^2 P_{2n-1} + 1^2.3^2\ldots(2n+1)^2\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n-1)^2}\right),$$

$$\therefore P_{2n+3} = (2n+1)^2(2n-1)^2\ldots 3^2 P_3$$
$$+ 1^2.3^2\ldots(2n+1)^2\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n-1)^2} + \ldots + \frac{1}{3^2}\right).$$ But $P_3 = 1$,

therefore $\quad P_{2n+3} = 1^2 . 3^2 ... (2n+1)^2 \left(\frac{1}{1^2} + \frac{1}{3^2} + ... + \frac{1}{(2n+1)^2}\right);$

$$\therefore \frac{P_{2n+3}}{1.2...(2n+3)} = \frac{1.3...(2n+1)}{2.4...(2n+2)} \cdot \frac{1}{2n+3} \left(\frac{1}{1^2} + \frac{1}{3^2} + ... + \frac{1}{(2n+1)^2}\right).$$

Hence $\quad \frac{(\sin^{-1} x)^3}{1.2.3} = \frac{1}{2} \cdot \frac{1}{1^2} \frac{x^3}{3} + \frac{1.3}{2.4} \left(\frac{1}{1^2} + \frac{1}{3^2}\right) \frac{x^5}{5} + ...$

4. Let $\quad \frac{x^4}{1.2.3.4} = P_4 \frac{\sin^4 x}{1.2.3.4} + P_6 \frac{\sin^6 x}{1.2...6} + ...,$

then by an exactly similar process we obtain

$$\frac{P_{2n+4}}{1.2...(2n+4)} = \frac{2.4...(2n+2)}{3.5...(2n+3)} \cdot \frac{1}{2n+4} \left(\frac{1}{2^2} + \frac{1}{4^2} + ... + \frac{1}{(2n+2)^2}\right),$$

therefore $\quad \frac{(\sin^{-1} x)^4}{1.2.3.4} = \frac{2}{3} \cdot \frac{1}{2^2} \cdot \frac{x^4}{4} + \frac{2.4}{3.5} \left(\frac{1}{2^2} + \frac{1}{4^2}\right) \frac{x^6}{6} + ...$

5. Let $\quad \frac{x^5}{1.2.3.4.5} = P_5 \frac{\sin^5 x}{1.2...5} + P_7 \frac{\sin^7 x}{1.2...7} + ...,$

then differentiating twice, and putting $\sin^{-1} x$ for x, we obtain

$$\frac{(\sin^{-1} x)^3}{1.2.3} = -x \left(P_5 \frac{x^4}{1.2.3.4} + P_7 \frac{x^6}{1.2...6} + ...\right) + (1-x^2) \left(P_5 \frac{x^3}{1.2.3} + ...\right).$$

But $\quad \frac{(\sin^{-1} x)^3}{1.2.3} = \frac{1}{2} \cdot \frac{1}{1^2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \left(\frac{1}{1^2} + \frac{1}{3^2}\right) \frac{x^5}{5} + ...;$

hence equating coefficients of x^{2n+3}, we have, after slight reduction,

$$P_{2n+5} = (2n+3)^2 P_{2n+3} + 1^2 . 3^2 ... (2n+1)^2 \left(\frac{1}{1^2} + \frac{1}{3^2} + ... + \frac{1}{(2n+1)^2}\right).$$

Denote $\frac{1}{1^2} + \frac{1}{3^2} + ... \frac{1}{(2n+1)^2}$ by C_{2n+1}, then we have

$$P_{2n+5} = (2n+3)^2 P_{2n+3} + 1^2 . 3^2 ... (2n+3)^2 . \frac{C_{2n+1}}{(2n+3)^2},$$

$$= (2n+3)^2 (2n+1)^2 P_{2n+1} + 1^2 . 3^2 ... (2n+3)^2 \left(\frac{C_{2n+1}}{(2n+3)^2} + \frac{C_{2n-1}}{(2n+1)^2}\right),$$

$$= (2n+3)^2 (2n+1)^2 ... 3^2 P_3$$
$$+ 1^2 . 3^2 ... (2n+3)^2 \left(\frac{C_1}{3^2} + \frac{C_3}{5^2} + ... + \frac{C_{2n+1}}{(2n+3)^2}\right).$$ But $P_3 = 0$,

$$\therefore \frac{P_{2n+5}}{1.2...(2n+5)} = \frac{1.3...(2n+3)}{2.4...(2n+4)} \cdot \frac{1}{2n+5} \left(\frac{C_1}{3^2} + \frac{C_3}{5^2} + ... + \frac{C_{2n+1}}{(2n+3)^2}\right),$$

$$\therefore \frac{(\sin^{-1} x)^5}{1.2.3.4.5} = \frac{1.3}{2.4} \cdot \frac{C_1}{3^2} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \left(\frac{C_1}{3^2} + \frac{C_3}{5^2}\right) \cdot \frac{x^7}{7} + ...$$

$$= \frac{1.3}{2.4} \cdot \frac{1}{1^2 . 3^2} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \left\{\frac{1}{1^2 . 3^2} + \frac{1}{5^2} \left(\frac{1}{1^2} + \frac{1}{3^2}\right)\right\} \frac{x^7}{7} + .$$

6. Let
$$\frac{x^6}{1.2\ldots6} = P_6\frac{\sin^6 x}{1.2\ldots6} + P_8\frac{\sin^8 x}{1.2\ldots8} + \ldots,$$

then by a similar process, if $C_{2n} = \frac{1}{2^2} + \frac{1}{4^2} + \ldots + \frac{1}{(2n)^2}$, we obtain

$$\frac{(\sin^{-1} x)^6}{1.2\ldots6} = \frac{2.4}{3.5}\cdot\frac{C_2}{4^2}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\left(\frac{C_2}{4^2} + \frac{C_4}{6^2}\right)\frac{x^8}{8} + \ldots$$

$$= \frac{2.4}{3.5}\cdot\frac{1}{2^2.4^2}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\left\{\frac{1}{2^2.4^2} + \frac{1}{6^2}\left(\frac{1}{2^2} + \frac{1}{4^2}\right)\right\}\frac{x^8}{8} + \ldots.$$

7. The forms which the expansions of $\frac{(\sin^{-1} x)^{2n-1}}{1.2\ldots(2n-1)}$ and $\frac{(\sin^{-1} x)^{2n}}{1.2\ldots(2n)}$ assume now become evident. Thus, let

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2n-1)^2} = \phi_1(2n-1),$$

$$\frac{1}{3^2}\phi_1(1) + \frac{1}{5^2}\phi_1(3) + \ldots + \frac{1}{(2n+1)^2}\phi_1(2n-1) = \phi_3(2n-1),$$

$$\frac{1}{5^2}\phi_3(1) + \frac{1}{7^2}\phi_3(3) + \ldots + \frac{1}{(2n+3)^2}\phi_3(2n-1) = \phi_5(2n-1), \text{ \&c.; then}$$

$$\frac{(\sin^{-1} x)^{2n-1}}{\Gamma(2n)} = \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}\phi_{2n-3}(1)\cdot\frac{x^{2n-1}}{2n-1}$$
$$+ \frac{1.3\ldots(2n-1)}{2.4\ldots2n}\cdot\phi_{2n-3}(3)\cdot\frac{x^{2n+1}}{2n+1} + \ldots\ldots\ldots\ldots(\text{I.})$$

Also if
$$\frac{1}{2^2} + \frac{1}{4^2} + \ldots + \frac{1}{(2n)^2} = \phi_2(2n),$$

$$\frac{1}{4^2}\phi_2(2) + \frac{1}{6^2}\phi_2(4) + \ldots + \frac{1}{(2n+2)^2}\phi_2(2n) = \phi_4(2n),$$

$$\frac{1}{6^2}\phi_4(2) + \frac{1}{8^2}\phi_4(4) + \ldots + \frac{1}{(2n+4)^2}\phi_4(2n) = \phi_6(2n), \text{ \&c.; then}$$

$$\frac{(\sin^{-1} x)^{2n}}{\Gamma(2n+1)} = \frac{2.4\ldots(2n-2)}{3.5\ldots(2n-1)}\phi_{2n-2}(2)\cdot\frac{x^{2n}}{2n}$$
$$+ \frac{2.4\ldots2n}{3.5\ldots(2n+1)}\cdot\phi_{2n-2}(4)\cdot\frac{x^{2n+2}}{2n+2} + \ldots\ldots\ldots\ldots(\text{II.})$$

It will be perceived that the property noticed by Professor Sylvester holds good in the above expansions. The formula (II.) may be derived from (I.) by augmenting in it every factor, every index, and every sub-index, by unity.

When $n=1$, the above formulæ (I.) and (II.) may appear to fail, as not giving immediately the expansions of $\sin^{-1} x$ and $\frac{(\sin^{-1} x)^2}{1.2}$. They are, however, perfectly correct, and we may obtain the required expansions from them as follows:—

$$\frac{(\sin^{-1}x)^{2n-1}}{\Gamma(2n)} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n)} \cdot \phi_{2n-3}(1) \cdot \frac{x^{2n-1}}{2n-1} + \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \cdot \phi_{2n-3}(3) \cdot \frac{x^{2n+1}}{2n+1} + \dots,$$

$$\frac{(\sin^{-1}x)^{2n}}{\Gamma(2n+1)} = \frac{1}{2}\left\{\frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})} \cdot \phi_{2n-2}(2) \cdot \frac{x^{2n}}{2n} + \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \cdot \phi_{2n-2}(4) \cdot \frac{x^{2n+2}}{2n+2} + \dots\right\},$$

$$\therefore (n=1),\ \frac{\sin^{-1}x}{\Gamma(2)} = \phi_{-1}(1) \cdot \frac{x}{1} + \frac{1}{2}\phi_{-1}(3) \cdot \frac{x^3}{3} + \frac{1.3}{2.4}\phi_{-1}(5) \cdot \frac{x^5}{5} + \dots (\alpha),$$

and $$\frac{(\sin^{-1}x)^2}{\Gamma(3)} = \phi_0(2) \cdot \frac{x^2}{2} + \frac{2}{3}\phi_0(4) \cdot \frac{x^4}{4} + \frac{2.4}{3.5}\phi_0(6) \cdot \frac{x^6}{6} + \dots (\beta).$$

Now $\phi_1(2n-1)\left(\text{which} = \frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n-1)^2}\right)$

$$= \frac{1}{1^2}\phi_{-1}(1) + \frac{1}{3^2} \cdot \phi_{-1}(3) + \dots + \frac{1}{(2n-1)^2}\phi_{-1}(2n-1).$$

Hence $\phi_{-1}(1)$, $\phi_{-1}(3)$, &c., are all equal to 1. Also, $\phi_2(2n)$ $\left(\text{which} = \frac{1}{2^2} + \frac{1}{4^2} + \dots + \frac{1}{(2n)^2}\right) = \frac{1}{2^2}\phi_0(2) + \frac{1}{4^2}\phi_0(4) + \dots + \frac{1}{(2n)^2}\phi_0(2n)$.

Hence $\phi_0(2)$, $\phi_0(4)$, &c., all $= 1$. Making these substitutions in (α) and (β), we obtain the expansions of $\sin^{-1}x$ and $\frac{(\sin^{-1}x)^2}{1.2}$.

NOTE.—In proposing Quest. 2236 (see *Reprint*, Vol. VII., p. 62), it did not occur to me that the formula (1.) is substantially the same as the well-known one $x = \sin x + \frac{1}{2} \cdot \frac{\sin^3 x}{3} + \text{\&c.}$ I would therefore wish to withdraw the statement, made at the end of Art. 1 of the solution, that the series there given for the value of π is probably new, which is not likely to be the case, since it may be obtained from the formula $x = \sin x + \text{\&c.}$, by making $x = \frac{1}{2}\pi$.

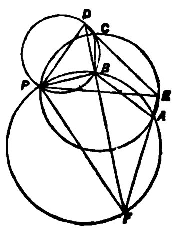
2441. (Proposed by F. A. TARLETON, M.A.)—Find the envelope of a conic which is circumscribed to a given triangle, and with respect to which two given lines are conjugate.

Solution by Professor HIRST.

Since a conic is cut harmonically by two conjugate lines, this Question is identical with the Question 1481, proposed by me and solved by Mr. GREER (*Reprint*, Vol. V., p. 85.)

2507. (Proposed by J. Wilson.)—Three circles pass through a point P; and PD, PE, PF are their diameters; if A, B, C be their other intersections, prove that the three triads of points EAF, FBD, DCE are collinear; and if a circle pass through the points P, D, E, F, the triad ABC is also collinear.

Solutions by Archer Stanley; R. Tucker, M.A.; *and others.*



1. Inverting the figure relative to the point P, we have the following well known and easily demonstrated theorems:—

"If D′E′F′ be the feet of the perpendiculars from a point P on the sides B′C, C′A′, A′B′ of a triangle, PE′A′F′, PF′B′D′, PD′C′E′ are three sets of concyclic points; and if moreover D′, E′, F′ are collinear, PA′B′C′ are also concyclic."

2. *Otherwise*:—The angle PAE = a right angle = PAF; therefore EAF is a straight line, and the like may be shown of FBD, DCE. Moreover if a circle pass through PDEF, the angles PDE, PFE are supplementary, and therefore PBC, PBA are also supplementary; hence, in this case, ABC will be a straight line.

[This Question might be stated more generally thus:—Three circles pass through a point P, and the chords PD, PE, PF subtend similar segments in the same direction, &c. According to a known theorem (See *Lady's and Gentleman's Diary*, Quest. 1934), if a point be taken on each of the sides of a triangle, the circles circumscribing the three triangles cut off by lines joining those points two and two, will intersect in one point; and the lines joining the common intersection to the points on the triangle make equal angles, in the same direction, with the sides; the evident converse of this gives the solution of the first part of the question. The second part, as stated in the question, was a Cambridge Senate-House Problem for 1838. (*Cambridge Mathematical Journal*, Vol. I., p. 168, quoted by Salmon, *Conics*, page 99, 4th Ed.; see also *Reprint*, Vol. VIII., p. 48, Quest. 2444). In the more general form, we have the following:—If any transversal cuts the sides BC, CA, AB of a triangle in a, b, c respectively, the circles circumscribing ABC, Abc, Bca, Cab intersect in a point, and conversely. If D be the common intersection, (DA, DB, DC), (DA, Db, Dc), (Da, DB, Dc), (Da, Db, DC) cut off similar segments (*Quarterly Journal of Mathematics*, Vol. IV., p. 362).

These properties are projective, and as the circular form somewhat disguises the symmetry, we subjoin the result of a more general view. Let two conics Q$abcd$, R$abcd$ be drawn. Take fixed points S, T on these respectively, and suppose that the sides of the triangle PQR pass successively through S, a, T, while the vertices Q, R move on Q$abcd$, R$abcd$ respectively. Then, observing four positions of P and the corresponding anharmonic ratio, it is clear that the locus of P is a conic passing through STbcd. Consequently, with reference to any triangle PQR, if we take points S on PQ, a on QR, T on RP, and draw through PST, QaS, RaT, respectively, three conics intersecting in the same two points elsewhere, they will have a third common intersection. To circles having similar segments upon three several chords, correspond conics with respect to which the points Pbcd, Qbcd, Rbcd subtend equal anharmonic ratios when taken in corresponding order.]

2559. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Four quadrics pass each through three of four arbitrary lines in space; show that the four poles with respect to them of any arbitrary plane are coplanar.

Solution by W. S. M'CAY, B.A.

The two lines in space which intersect the four arbitrary lines being common generators of the four quadrics, and the two planes determined by them with the line connecting their two points of intersection with the arbitrary plane being consequently tangent planes to the four surfaces, therefore the fourth plane, harmonically conjugate to the arbitrary with respect to the two latter planes, contains the four poles in question, and therefore, &c.

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

Solution by T. COTTERILL, M.A.; REV. R. TOWNSEND, F.R.S.; *and others.*

Let Z be the intersection of two conjugate chords AA′, BB′ of a conic; then X, the pole of AA′, is on BB′; XA, XA′ are tangents, and XZ is harmonic to BB′. The anharmonic ratio of any point on the conic to the points AA′BB′ being the same as that of A (AA′BB′) or (XZBB′), is consequently harmonic. But in the quadrilateral, if two lines divide harmonically the three diagonals, they are conjugates to all the inscribed conics, and consequently, from what has been said, cut each conic in four points harmonic to the points upon it.

2498. (Proposed by R. BALL, M.A.)—Let S_n represent the sum of the nth powers of the roots of the cubic $ax^3+3bx^2+3cx+d=0$, and S'_n the result of interchanging a and d, b and c, in S_n; then prove that

$$a^{2n}S_{2n} = (a^nS_n)^2 + 2(-1)^{n+1}a^nd^nS'_n,$$

and hence show that

$$a^{12}S_{12} = (3a^4d^2 - 108a^3bcd - 54a^3c^3 + 729a^2b^2c^2 + 162a^2b^3d - 1458ab^4c + 729b^6)^2$$
$$-2a^6(3d^4a^2 - 108d^3cba - 54d^3b^3 + 729b^2c^2d^2 + 162d^2c^3a - 1458dc^4b + 729c^6).$$

Solution by SAMUEL ROBERTS, M.A.

If we represent by Π_1 the operator $3b\dfrac{d}{da} + 2c\dfrac{d}{db} + d\dfrac{d}{dc}$, and by Π_2 the

operator $3c\frac{d}{dd}+2b\frac{d}{dc}+a\frac{d}{db}$, the sum of the nth powers of roots of the given equation will be given by $(-)^n \frac{\Pi_1^{\,n}}{[n-1]}\log(a)$, and the sum of the inverse nth powers will be given by $(-)^n \frac{\Pi_2^{\,n}}{[n-1]}\log(d)$. But

$$\alpha^{2n}+\beta^{2n}+\gamma^{2n}=(\alpha^n+\beta^n+\gamma^n)^2-2\alpha^n\beta^n\gamma^n\left(\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}\right),$$

and therefore we have the first result in the question.

As a particular case $\quad a^{12}S_{12}=a^{12}\left(\frac{\Pi_1^{\,6}}{[5]}\log a\right)^2-2a^6d^6\left(\frac{\Pi_2^{\,6}}{[5]}\log d\right);$

and performing the operations indicated, we get the second result. We may either avail ourselves of Arbogast's rules of abbreviation, or apply the general formula by making $n=3$.

2420. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—A set of three numbers, each of them not exceeding a number n, being arbitrarily taken from a list of such combinations, let p_n denote the probability that they shall form the sides of a possible triangle; and show that, provided n be an even number, the consecutive probabilities p_n, p_{n+1}, p_{n+2} are in arithmetical progression.

Solution by the PROPOSER.

1. Assuming that repetitions are allowed, the total number of combinations contained in the list from which the set of numbers is taken is $\frac{n(n+1)(n+2)}{2\,.\,3}$; and if u denote the number of cases which exhibit the sides of a possible triangle, the sought probability will be

$$p=\frac{2\,.\,3u}{n(n+1)(n+2)}.$$

To determine the form of u, let n be supposed to become $n+1$ and the number of new favourable cases will express the value of the finite difference Δu. These combinations will all of them necessarily contain the new number $n+1$, which may be conveniently placed first as the highest in the order of magnitude. To obtain a possible triangle, in every case the sum of the other two numbers which follow must exceed the first number $n+1$. Hence, giving higher numbers always the precedence, if the second number of the triad be $n+1$ it may be associated with any one of the consecutive numbers from $n+1$ down to 1 inclusive. Also if the second number of a triad be n, it may be associated with numbers from n down to 2; if it be $n-1$ it may be taken with numbers from $n-1$ down to three inclusive, &c.

The numbers of cases thus presented are respectively $n+1$, $n-1$, $n-3$, &c.; and collecting them we get

$$\Delta u = (n+1)+(n-1)+(n-3)+(n-5)+\&c,$$

including all positive terms.

By reversing the order of the terms, we get

$$n \text{ even, } \Delta u = 1+3+5\ldots\ldots+(n+1) = \tfrac{1}{4}(n^2+4n+4),$$
$$n \text{ odd, } \Delta u = 2+4+6\ldots\ldots+(n+1) = \tfrac{1}{4}(n^2+4n+3).$$

Therefore, generally, $\Delta u = \frac{1}{8}\left\{2n^2+8n+7+(-1)^n\right\}$,

and by integration, making u vanish with x, we obtain

$$u_n = \frac{4n^3+18n^2+20n+3-3(-1)^n}{48}, \text{ and } p = \frac{4n^3+18n^2+20n+3-3(-1)^n}{8n(n+1)(n+2)}.$$

By separating the cases of n odd and even the expression becomes simplified in each. Thus

$$n \text{ even, } p = \frac{4n^3+18n^2+20n}{8n(n+1)(n+2)} = \frac{2n+5}{4(n+1)}$$
$$n \text{ odd, } p = \frac{4n^3+18n^2+20n+6}{8n(n+1)(n+2)} = \frac{(2n+1)(n+3)}{4n(n+2)}.$$

Hence if, according to the question, n be an even number we shall have

$$p_n = \frac{2n+5}{4(n+1)},\quad p_{n+1} = \frac{(2n+3)(n+4)}{4(n+1)(n+3)},\quad p_{n+2} = \frac{2n+9}{4(n+3)},$$

which are in a decreasing arithmetical progression, the common difference, or decrement, being $\dfrac{3}{4(n+1)(n+3)}$.

2. If repetitions are not admissible, the total number of combinations in the list from which the set of numbers is taken will then be $\dfrac{n(n-1)(n-2)}{2.3}$.

In this case, the new number $n+1$ being again placed first, if the second number of the triad be n, the third may be any number from $n-1$ down to 2; if the second number be $n-1$, it may be associated with numbers from $n-2$ down to 3, &c. The numbers of cases thus arising are severally $n-2$, $n-4$, &c.

Hence, collecting them, we have

$$n \text{ even, } \Delta u' = 2+4+6\ldots+n-2 = \tfrac{1}{4}(n^2-2n),$$
$$n \text{ odd, } \Delta u' = 1+3+5\ldots+n-2 = \tfrac{1}{4}(n^2-2n+1).$$

Therefore, generally, $\Delta u' = \frac{1}{8}\left\{2n^2-4n+1-(-1)^n\right\}$,

$$u'_n = \frac{4n^3-18n^2+20n-3+3(-1)^n}{48}, \text{ and } p' = \frac{4n^3-18n^2+20n-3+3(-1)^n}{8n(n-1)(n-2)}.$$

This expression gives,

$$\text{for } n \text{ even, } p' = \frac{4n^3-18n^2+20n}{8n(n-1)(n-2)} = \frac{2n-5}{4(n-1)},$$
$$\text{for } n \text{ odd, } p' = \frac{4n^3-18n^2+20n-6}{8n(n-1)(n-2)} = \frac{(2n-1)(n-3)}{4n(n-2)}.$$

Hence, if n be even, we have

$$p'_n = \frac{2n-5}{4(n-1)}, \quad p'_{n-1} = \frac{(2n-3)(n-4)}{4(n-1)(n-3)}, \quad p'_{n-2} = \frac{2n-9}{4(n-3)},$$

which are likewise in arithmetical progression, the common difference being

$$\frac{3}{4(n-1)(n-3)}.$$

Note.—The values of the probability p in the two cases are mutually interchangeable by employing $-n$ instead of n.

It is further remarkable that

$$u_n + u_{n-1} = \frac{n(n+1)(n+2)}{2.3}, \quad u'_n + u'_{n+1} = \frac{n(n-1)(n-2)}{2.3},$$

which determine the formulæ

$$p_n = 1 - \frac{n-1}{n+2} p_{n-1}, \quad p'_n = \frac{n-3}{n}(1 - p'_{n-1}).$$

II. *Solution by* STEPHEN WATSON.

If in the series, n being even, $n, n-1, n-2, \ldots . 4, 3, 2, 1 \ldots\ldots\ldots (1)$, we take n as one number; $n-1$, $n-2$, successively, as another; and then in each case as many of the others as will form a triangle; we get the numbers of favourable cases as in the first line below. Next take $n-1$; and successively $n-2$, $n-3$: then in the same way we have the second line below; proceeding thus, we have

$$\begin{array}{l}
n-3 \ldots\ldots + 5 + 3 + 1 \\
n-4 \ldots\ldots + 4 + 2 \\
n-5 \ldots\ldots + 3 + 1 \\
n-6 \ldots\ldots + 2 \\
\ \ \vdots \quad \ldots\ldots + 1 \\
\ \ 2 \\
\ \ 1
\end{array}$$

Hence, adding vertically, we have the sum

$$= \tfrac{1}{2}(n-2)(n-3) \ldots + 3.5 + 2.3 + 1 = \tfrac{1}{24} n (n-2)(2n-5).$$

In like manner, when n is odd, the number of possible triangles is

$$\tfrac{1}{24}(n-1)(n-3)(2n-1).$$

Also the total number of ways in which three numbers can be taken is $\frac{1}{6}n(n-1)(n-2)$; hence, according as n is even or odd,

$$p_n = \frac{2n-5}{4(n-1)}, \qquad \therefore\ p_{n+2} = \frac{2n-1}{4(n+1)};$$

$$\text{or}\quad p_n = \frac{(n-3)(2n-1)}{4n(n-2)}, \qquad \therefore\ p_{n+1} = \frac{(n-2)(2n+1)}{4(n^2-1)} \ \text{(here } n \text{ is even).}$$

Therefore, when n is even, p_n, p_{n+1}, p_{n+2} are in arithmetical progression, for

$$p_{n+1} - p_n = p_{n+2} - p_{n+1} = \frac{3}{4(n^2-1)}.$$

[In solution the three numbers taken at random are supposed to be all different.]

2549. (Proposed by J. GRIFFITHS, M.A.)—Solve the simultaneous equations $yz+x=14,\ zx+y=11,\ xy+z=10.$

Solutions by R. TUCKER, M.A.; S. BILLS; *and many others.*

1. Let $P^3 = aP^2+bP-c = 0$ be the equation whose roots are x, y, z; then

$$a+b=35 \quad\text{(i.)}$$
$$b-3c+ab+ac=404 \quad\text{(ii.)}$$
$$c+b^2-2ac+a^2c-2bc+c^2=1510 \quad\text{(iii.)}$$

These equations reduce to

$$c^2+(a^2-69)\,c+a^2-70a-315=0 \quad(\alpha),$$
$$(3-a)\,c+a^2-34a+369=0 \quad(\beta);$$

whence we obtain

$$a^5-35a^4+258a^3+3454a^2-56331a+209709=0,$$

or, $$(a-9)\,(a^4-26a^3+24a^2+3670a+23301)=0.$$

Hence one set of answers is found from

$$a=9,\ b=26,\ c=24,$$

and $$P^3-9P^2+26P-24=(P-2)(P-3)(P-4)=0;$$

therefore $$x=2,\ y=3,\ z=4.$$

2. *Otherwise:* from the 2nd and 3rd equations we have

$$y=\frac{10x-11}{x^2-1},\quad z=\frac{11x-10}{x^2-1};$$

substituting these values in the 1st equation, it becomes

$$x^5-14x^4-2x^3+138x^2-220x+96=0,$$

or $$(x-2)(x^4-12x^3-26x^2+86x-48)=0;$$

whence $x=2$, and therefore $y=3$ and $z=4$.

2362. (Proposed by R. TUCKER, M.A.)—Find the curves whose circles of curvature intercept upon a given line portions which (1) vary as the radii of curvature, (2) are constant.

Solution by the PROPOSER.

1. Taking the fixed line as axis of x, we have in the *first case* (since the intercept varies as the radius) the angle at the centre constant, and therefore β (the ordinate of the centre of curvature) varies as ρ.

$$\text{Let}\quad c\rho=\beta,\ \text{or}\quad \frac{q}{(1+p^2)\left\{c(1+p^2)^{\frac12}-1\right\}}=\frac{1}{y},$$

therefore $$\left\{\frac{c\,(1+p^2)^{\frac12}+1}{1+p^2}-\frac{c^2}{c\,(1+p^2)^{\frac12}-1}\right\}qp=-\frac{p}{y},$$

whence we get

$$c\,(1+p^2)^{\frac12}+\log\,(1+p^2)^{\frac12}+\log\left(\frac{y}{a}\right)=c\,(1+p^2)+\log\left\{c\,(1+p^2)^{\frac12}-1\right\}$$

or $(c-y)(1+p^2)^{\frac{1}{2}} = a$, which gives the circle $(x+b)^2+(y-c)^2 = a^2$, where a, b, c are constants.

2. In the *second case*, we obtain

$$\rho^2 - \beta^2 = k^2$$

if $2k$ is the constant intercept; which leads to the differential equation

$$q^2(k^2+y^2) + 2yq(1+p^2) = p^2(1+p^2)^2,$$

of which the circle is a solution, as may be seen otherwise.

No other curves apparently can be found fulfilling the conditions.

1920. (Proposed by C. W. MERRIFIELD, F.R.S.)—What is the condition that an equation should have a pair of roots equal in magnitude and opposite in sign?

I. *Solution by* F. D. THOMSON, M.A.; S. BILLS; W. H. LAVERTY; *and others.*

Let $F(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0 \ldots\ldots\ldots\ldots (1)$,

be the proposed equation, of which a, $-a$ are roots; then

$$F(a) = 0,\ F(-a) = 0;$$

therefore $\quad F(a) + F(-a) = 0 \ldots\ldots (2),\ F(a) - F(-a) = 0 \ldots\ldots (3).$

First let n be even and $= 2m$; then from (2) and (3), writing α for a^2,

$$\alpha^m + p_2\alpha^{m-1} + p_4\alpha^{m-2} + \ldots + p_n = 0 \ldots\ldots\ldots\ldots (4),$$

$$p_1\alpha^{m-1} + p_3\alpha^{m-2} + \ldots + p_{n-1} = 0 \ldots\ldots\ldots\ldots (5);$$

and the result of eliminating α between the equations (4) and (5) will be the condition required.

If n be odd and $= 2m+1$, we obtain in like manner

$$\alpha^m + p_2\alpha^{m-1} + \ldots + p_{n-1} = 0 \ldots\ldots\ldots\ldots (6),$$

$$p_1\alpha^m + p_3\alpha^{m-1} + \ldots + p_n = 0 \ldots\ldots\ldots\ldots (7);$$

and we have to eliminate α between (6) and (7).

II. *Solution by the* PROPOSER.

Let $\pm a$ be the pair of roots; then the equation must have $x^2 - a^2$ as a factor. Now if we multiply this quantity (x^2-a^2) by an expression of the form

$$q_0x^{n-2} + q_1x^{n-3} + \ldots\ldots\ldots + q_{n-2} \ldots\ldots\ldots\ldots (1),$$

it is clear that in the resulting expression

$$p_0x^n + p_1x^{n-1} + \ldots\ldots\ldots + p_n \ldots\ldots\ldots\ldots (2),$$

we shall get the odd terms (whose coefficients are of the form p_{2m}) from the odd terms (q_{2m-2}) of the expression (1) without regard to the even terms of either expression; and, similarly, that we shall get the even terms of (1) from the even terms of (2), independently of the odd terms.

The condition therefore is, that

$$p_0 x^n + p_2 x^{n-2} + \ldots\ldots, \text{ and } p_1 x^{n-1} + p_3 x^{n-3} + \ldots\ldots$$

should have a common measure.

III. *Solution by the* REV. J. L. KITCHIN, M.A.

1. Let $f(x) = 0 = (x-a_1)(x+a_1)(x-a_2)\ldots\ldots\ldots(x-a_n)$

therefore $f(-x) = (-1)^n (x-a_1)(x+a_1)(x+a_2)\ldots\ldots(x+a_n)$,

therefore $f(x) = 0, f(-x) = 0$, have a common measure $x^2-a_1^2$.

Hence the most practicable condition for the roots in question is that found by writing $-x$ in the equation, and then seeking their Greatest Common Measure. The Greatest Common Measure will, if the remaining roots are unequal, be the quadratic factor containing the roots in question.

2. Let $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0 \ldots\ldots(1)$;

assume $x^2 - a_1^2$ as the factor which put equal to 0 gives the roots equal but of opposite sign, and

$$x^{n-2} + q_1 x^{n-3} + q_2 x^{n-4} + \ldots\ldots + q_{n-4}x^2 + q_{n-3}x + q_{n-2} = 0$$

the quotient of (1) by $x^2 - a_1^2 = 0$. From these last, we get

$$x^n + q_1 x^{n-1} + (q_2 - a_1^2)\, x^{n-2} + (q_3 - a_1^2 q_1)\, x^{n-3} + (q_4 - a_1^2 q_2)\, x^{n-4} + \ldots\ldots$$
$$+ (q_{n-2} - a_1^2 q_{n-4})\, x^2 - q_{n-3}a_1^2\, x - a_1^2 q_{n-2} = 0.$$

Whence comparing with (1), we get

$$p_1 = q_1,\ p_2 = q_2 - a_1^2,\ p_3 = q_3 - a_1^2 q_1,\ p_4 = q_4 - a_1^2 q_2 \ldots\ldots\ldots$$
$$p_{n-2} = q_{n-2} - a_1^2 q_{n-4},\ p_{n-1} = -a_1^2 q_{n-3},\ p_n = -a_1^2 q_{n-2};$$

$$\text{therefore } p_2 - q_2 = \frac{p_3 - q_3}{p_1} = \frac{p_4 - q_4}{q_2} = \frac{p_{n-2} - q_{n-2}}{q_{n-4}} = \frac{p_{n-1}}{q_{n-3}} = \frac{p_n}{q_{n-2}}.$$

Here there are $n-2$ equations, and $n-3$ unknown quantities, so their elimination will tend to an equation in $p_1, p_2 \ldots p_n$, which is the condition sought.

If the equation be a cubic, we get at once $p_1 p_2 = p_3$ as the condition.

In the same manner we find for a biquadratic the condition,

$$p_1 p_2 p_3 = p_1^2 p_4 + p_3^2, \text{ and so on.}$$

As the equation increases in degree the condition becomes more and more complicated; in fact the simple device of changing x into $-x$ and seeking the Greatest Common Measure of the original and resulting equations is infinitely preferable as a means of discovering whether a given equation contains two roots equal, but opposite in sign, with the single exception of the cubic.

2367. (Proposed by F. A. TARLETON, M.A.)—If $x^n + b_1x^{n-1} + b_2x^{n-2} + \dots + b_n = 0$ be an algebraical equation whose roots are $x_1, x_2, \dots x_n$ show that

$$\begin{vmatrix} \frac{db_1}{dx_1}, & \frac{db_2}{dx_1}, & \frac{db_3}{dx_1} & \cdots & \frac{db_n}{dx_1} \\ \frac{db_1}{dx_2}, & \frac{db_2}{dx_2}, & \frac{db_3}{dx_2} & \cdots & \frac{db_n}{dx_2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{db_1}{dx_n} & \cdots & \cdots & \cdots & \frac{db_n}{dx_n} \end{vmatrix} = \sqrt{(\Delta)},$$ where Δ is the discriminant.

Solution by J. DALE; *and* R. W. SYMES, B.A.

Neglecting the sign for the present, we have

$$b_1 = \Sigma x_1,\ b_2 = \Sigma x_1x_2,\ \dots\ b_n = \Sigma x_1x_2 \dots x_n;$$

$$\frac{db_1}{dx_1} = \frac{db_1}{dx_2} = \frac{db_1}{dx_3} = \dots\dots\dots = \frac{db_1}{dx_n} = 1;$$

$$\frac{db_2}{dx_1} = b_1 - x_1,\ \frac{db_2}{dx_2} = b_1 - x_2,\ \dots\dots\ \frac{db_2}{dx_n} = b_1 - x_n;$$

$$\frac{db_3}{dx_1} = b_2 - b_1x_1 + x_1^2;\ \frac{db_3}{dx_2} = b_2 - b_1x_2 + x_2^2;\ \dots\dots$$

$$\frac{db_m}{dx_n} = b_{m-1} - b_{m-2}x_n + \dots\dots\dots \pm (b_1x_n^{m-2} - x_n^{m-1}).$$

Substituting these values in the determinant, it becomes, after reduction,

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \dots\dots x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 \dots\dots x_2^{n-1} \\ 1 & x_3 & x_3^2 & x_3^3 \dots\dots x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots \cdots\cdots \vdots \\ 1 & x_n & x_n^2 & x_n^3 \dots\dots x_n^{n-1} \end{vmatrix}$$

Subtracting the last line from each of the others, dividing out, and repeating the operation, the determinant becomes

$$(x_1 - x_2) \dots (x_1 - x_n)\ (x_2 - x_3) \dots (x_2 - x_n) \dots (x_{n-1} - x_n);$$

therefore the square of the determinant is equal to the last or absolute term in the equation to the squares of the differences of the given equation, that is, to the discriminant of the given equation. Hence the theorem is proved.

[The method used by Dr. SALMON (*Higher Algebra*, p. 13, ex. 5) for finding the value of the determinant to which Mr. DALE reduces the question

is immediately applicable to the determinant in the question itself. Thus we see at once that, if two roots are equal, the determinant vanishes; hence $(x_1-x_2)\ldots(x_{n-1}-x_n)$ is a factor. Also, since a root appears only in the first degree, and in $n-1$ of the rows, the order in which a root appears in the worked out result is $n-1$, by a known rule. Hence there can only be a numerical factor besides, and this is proved to be unity by observing that if $x_n = x_{n-1} = \ldots = x_3 = 0$, the numerical multiplier is not altered; thus, if M be the multiplier, we have $Mx_1x_2(x_1-x_2) = x_1x_2(x_1-x_2)$.]

1934. (Proposed by M. W. Crofton, B.A.) — Two bags contain m and n balls respectively; an arbitrary number is drawn from each, 0 being considered a number [and all numbers being supposed equally probable]: find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$.

Solution by Stephen Watson.

Let m be $> n$, and r the assigned number, and take the question in the following cases:—

1. When r is not $> n$. In this case the number of ways favourable to the chance is the number of ways in which any two of the numbers 0, 1, 2, ..., each figure being taken twice, can be added so as to give r, and is $r+1$.

2. When $r > n$ but not $> m$. Here the favourable cases are $n+1$.

3. When $r > m$. The favourable cases are $n-\{r-(m+1)\}$. Hence, $(m+1)(n+1)$ being the total cases, the chances in the three cases are

$$\frac{r+1}{(m+1)(n+1)},\quad \frac{1}{m+1},\quad \frac{n-\{r-(m+1)\}}{(m+1)(n+1)}.$$

[If we denote these functions by F_1, F_2, F_3, respectively, we have

$$F_1(m,n,r)+F_3(m,n,r) = F_2(m,n,r)+F_2(n,m,r).$$

See *Reprint*, Vol. IV., p. 86.]

From these the respective chances, of some number not greater than n, of some number $> n$ but not $> m$, and of some number $> m$ but not $> m+n$, are easily found to be

$$\frac{\frac{1}{2}n+1}{m+1},\quad \frac{m-n}{m+1},\quad \frac{\frac{1}{2}n}{m+1};$$

and the sum of these is unity, as it ought to be.

[In the above solution, to which the amended enunciation of the question is adapted, it is supposed that all possible numbers have severally an equal chance of being drawn from each bag. The solution may also be regarded as a strictly accurate one to the following question:—

Two bags contain $m+1$ and $n+1$ counters, numbered 0, 1, 2 ... m, and 0, 1, 2 ... n, respectively; a counter is drawn from each; find the chance of the total number drawn being equal to any assigned integer, from 0 to $m+n$.

If we take the problem in its general and unrestricted form, and suppose that balls are drawn at random from each bag, that is to say, that all the possible ways in which a ball or balls can be drawn are equally probable, the chance of drawing any given number r in the manner stated in the question will evidently be unchanged if a single drawing be made from one bag containing all the $m+n$ balls; and its value will therefore be the coefficient of x^r in $\left(\frac{1+x}{2}\right)^{m+n}$.]

NOTE ON QUESTION 2471: BY W. S. B. WOOLHOUSE, F.R.A.S.

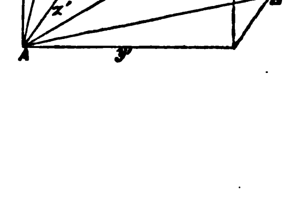
According to one of the formulæ stated in Question 2471 [*Reprint*, Vol. VIII., p. 103], the average square of the area of a triangle, formed by joining three arbitrary points, is $= \frac{3}{2} h^2 k^2$. This simple and curious relation applies equally when the three points are taken on any curve line, or on any kind of line whatever, if h, k still denote the radii of gyration of the same with respect to the principal axes passing through the centre of gravity.

It may also be observed that in the general question, as proposed, the formulæ which express the values of (Δ) and (Δ^2) are free from any condition as to convexity of boundary. For both of these formulæ the form of boundary is absolutely unrestricted. But the subsistence of the subsequent formulæ, which express the values of the several probabilites, requires the condition of a convex boundary, since in the ulterior reasoning upon which they are founded in the solution of the question, it is tacitly implied that the several straight lines which connect the points lie wholly upon the given surface or within the given boundary.

[The substance of this note was intended to be annexed to the solution, but was accidentally omitted.]

1496. (Proposed by M. COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are ABCD, and a parallelepiped formed from it as follows, have the same centre of gravity; viz. through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and DB; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

Solution by J. J. WALKER, M.A.



The theorem proposed for proof in this question coincides with one to which I was led in considering "some analogues in space to properties of the parallelogram" (see *Messenger of Mathematics*, Vol. IV., p. 144), inasmuch as the construction directed gives a parallelepiped related to the tetrahedron thus: the vertices BCD are the corners opposite to A of the three parallelogram-faces of the parallelepiped which cointersect at A. Call the three edges of

the parallelepiped (x', y', z'). Then the equation to the plane BCD referred to the three edges cointersecting in A will be $xy'z'+yx'z'+zx'z'-2x'y'z'=0$, since this equation is evidently satisfied by the coordinates $(0, y', z')$, $(x', 0, z')$, $(x', y', 0)$ of BCD. Also the plane passing through the centre of gravity of the tetrahedron, and parallel to BCD, will have as its equation

$$2xy'z'+2yz'x'+2zx'y'-3x'y'z'=0,$$

(since this plane intercepts on the axes lengths which are $\frac{3}{2}$ of those intercepted by the plane BCD,) and will evidently pass through the point $(\frac{1}{2}x', \frac{1}{2}y', \frac{1}{2}z')$, that is, through the centre of gravity of the parallelepiped. Similarly planes drawn through the centre of gravity of the tetrahedron, and parallel to the other three faces, may be shown to pass through that of the parallelepiped. Hence these two points coincide.

[The centre of gravity of the pyramid is the centre of gravity of four equal particles at its corners, and therefore bisects the straight line joining the middle points of AB, CD; it coincides, therefore, with the centre of gravity of the parallelepiped. Another solution is given on p. 87 of Vol. VIII. of the *Reprint*.]

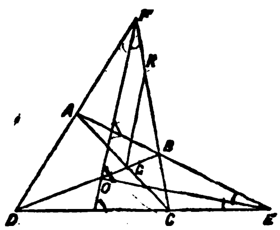
2411. (Proposed by Professor SYLVESTER.)—

1. Prove geometrically that the quadrangle formed by the intersections of one pair of right lines equally inclined at any angle to a given direction with another pair equally inclined at any other angle thereto, is inscribable in a circle, and that the third pair of sides of such quadrangle will also be equally inclined to the same direction.

2. Hence obtain, as an instantaneous algebraical inference, the theorem that the intersection of any two right cones with parallel axes is a spherical curve contained in a known sphere and a known third right cone whose axis is parallel to those of the given two.

Solution (1) *by the* REV. J. WOLSTENHOLME, M.A.; (2) *by the* PROPOSER.

1. Let the quadrangle be ABCD; and let FO, EO be straight lines bisecting the angles E, F, and by the conditions of the question at right angles to each other. Then we have



$$\angle ABC = B\hat{F}O + F\hat{O}E + O\hat{E}B = \tfrac{1}{2}F + O + \tfrac{1}{2}E,$$

and $$\angle ADC = F\hat{O}E - O\hat{E}D - DFO = O - \tfrac{1}{2}F - \tfrac{1}{2}E,$$

$\therefore\ \angle ABC + \angle ADC = \text{twice } O$
(this is general) $\quad = \text{two right angles}$;
or A, B, C, D lie on a circle.

If AC, BD meet in G, and GK bisecting the angle AGB meet FB in K,

$$2\angle GKB = 2DBC - AGB = DBC - ACB = DAC - ACB = DFC,$$

therefore $\angle GKB = \frac{1}{2}DFC = OFC$; or GK is parallel to OF.

[Mr. Wolstenholme remarks that he set this problem in the year 1859 at Christ's College, Cambridge. See also the solution of Quest. 1950, *Reprint*, Vol. V., p. 105.]

2. We now see that if

$$F = (x-a)^2 - m(z-c)^2, \text{ and } G = (x-a)^2 - m'(z-c')^2,$$

there must exist a ratio $p : q$ such that

$$\frac{pF+qG}{p+q} = H = (x-a'')^2 - m''(z-c'')^2,$$

where the three pairs of lines F, G, H will intersect in the angles of a quadrangle inscribable in a circle, and that accordingly another ratio $r : s$ may be assumed such that

$$\frac{rF+sG}{r+s} = K = (x-\alpha)^2+(z-\gamma)^2 = C.$$

Hence it follows that

$$\frac{p(F+y^2)+q(G+y^2)}{p+q} = H+y^2, \text{ and } \frac{r(F+y^2)+s(G+y^2)}{r+s} = K+y^2;$$

so that, if $(x-a)^2+y^2 = m(z-c)^2 \ldots(1)$ and $(x-a')^2+y^2 = m'(z-c')^2 \ldots(2)$,

we must also have $\qquad (x-\alpha)^2+y^2+(z-\gamma)^2 = C \ldots\ldots\ldots\ldots\ldots\ldots (3)$,

and $\qquad (x-a'')^2+y^2 = m''(z-c'')^2 \ldots\ldots\ldots\ldots\ldots\ldots (4)$,

which proves the theorem, and shows how the containing third cone and sphere may be geometrically determined.

The mere existence of a containing sphere and third cone, or several such (real or imaginary), might of course also be inferred *à priori* from an enumeration of the disposable constants by aid of which (1) and (2) admit of being brought into linear equivalence with (3) and with (4). There will in each case be four such disposable constants $\frac{p}{q}$, α, γ, C; or $\frac{p}{q}$, a'', c'', m''; to satisfy four equations arising out of the coefficients of z^2, z, x, 1.

2573. (Proposed by Professor CAYLEY.)—The envelope of a variable circle having for its diameter the double ordinate of a rectangular cubic is a Cartesian. [DEF.—The expression "a rectangular cubic" is used to express a cubic with three real asymptotes, having a diameter at right angles to one of the asymptotes and at an angle of 45° to each of the other two asymptotes, viz., the equation of such a cubic is $xy^2 = x^3+bx^2+cx+d$.]

Solution by the PROPOSER.

The equation of the variable circle may be taken to be

$$(x-\theta)^2+y^2 = \theta^2-2m\theta+\alpha+\frac{2A}{\theta},$$

viz., θ being the abscissa of the rectangular cubic, the squared ordinate is taken to be $=\frac{1}{\theta}(\theta^3-2m\theta^2+\alpha\theta+2A)$, or, what is the same thing, the equation of the variable circle is $\qquad x^2+y^2-\alpha-2(x-m)\theta-\frac{2A}{\theta} = 0.$

Hence, taking the derived equation in regard to θ, we have

$$x-m-\frac{A}{\theta^2}=0, \text{ and thence } x^2+y^2-a=\frac{4A}{\theta};$$

therefore $$(x^2+y^2-a)^2=\frac{16A^2}{\theta^2}=16A(x-m);$$

that is, the equation of the envelope is

$$(x^2+y^2-a)^2-16A(x-m)=0,$$

which is a known form of the equation of a Cartesian.

[Mr. WOLSTENHOLME obtains the equation of the envelope in the equivalent form

$$(x^2+y^2-c)^2=4d(b+2x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and proceeds with his solution as follows:—

The axis of x divides (1) symmetrically, and if it be a Cartesian its foci will be in this axis: let two of these be at distances α, β from the origin, and let the equation of the Cartesian be

$$m\left\{(x-\alpha)^2+y^2\right\}^{\frac{1}{2}}+n\left\{(x-\beta)^2+y^2\right\}^{\frac{1}{2}}=1.$$

This, on developing and putting $m^2\alpha=n^2\beta$, becomes

$$\left\{x^2+y^2-\frac{m^2\alpha^2}{n^2}-\frac{m^2+n^2}{(m^2-n^2)^2}\right\}^2=\frac{-8m^2\alpha x}{(m^2-n^2)^2}+\frac{4m^2\alpha^2(m^2+n^2)}{n^2(m^2-n^2)^2}+\frac{4m^2n^2}{(m^2-n^2)^4}\ldots(2),$$

which coincides with (1) if

$$c=\frac{m^2\alpha^2}{n^2}+\frac{m^2+n^2}{(m^2-n^2)^2},\quad d=\frac{-m^2\alpha}{(m^2-n^2)^2},\quad -b=\frac{\alpha(m^2+n^2)}{n^2}+\frac{n^2}{\alpha(m^2-n^2)^2}.$$

These give immediately the equation for α, viz. $\alpha^3+b\alpha^2+c\alpha+d=0$, a cubic whose three roots give the positions of the three foci.]

2358. (Proposed by Professor CREMONA.)—La condition qu'une conique divise harmoniquement les trois diagonales d'un quadrilatère circonscrit à une autre conique, coincide avec la condition que la première conique soit circonscrite à un triangle conjugué à l'autre.

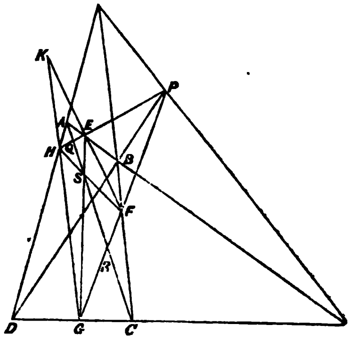
Solution by T. COTTERILL, M.A.

If a conic U cut the diagonals of a quadrilateral harmonically, the three pairs of lines through its intersections with two diagonals are evidently conjugates to a conic Σ inscribed in the quadrilateral. But if two lines are conjugate to a conic, the polar to the conic of any point on one of the lines cuts the lines in a pair of points harmonics to the points in which it meets the conic. Hence the polar to Σ of a point of intersection of U with one of the two diagonals cuts Σ in two points conjugates to its intersections with the three pairs of lines; and, by a well known theorem, the same points on

Σ are conjugates to the intersections of the line with U. But if a line cuts two conics U and Σ harmonically, and the pole of the line to Σ is on U, then this pole and the two points of intersection of the line and U form a triangle conjugate to Σ.

NOTE ON QUESTION 2478: BY THE REV. J. WOLSTENHOLME, M.A.

The distribution of ten points into ten rows of three in a row may be very simply done by the following construction:— Take any quadrangle ABCD, and from a point P on the intersection of two diagonals draw any two straight lines, meeting the third diagonal in Q, R, and the sides in E, F, G, H. Then (PEHQ) and (PGFR) are both harmonic ranges. Therefore FH and EG will meet in QR, in S suppose. Then the ten points A, B, C, D, E, F, G, H, P, S are ten points in the ten rows AHD, ASC, AEB, PBD, BFC, CGD, HEP, ESG, FSH, PFG.



If EF and GH intersect in K, then the above-mentioned ten points and K form eleven points in twelve rows of three.

[Professor SYLVESTER remarks that the general theorem in Quest. 2572 shows that with 11 points $\frac{1}{6}(10 \cdot 9)$, *i.e. fifteen*, rows of three may be formed, such points being pluperfect points of the 11th order lying on a cubic curve.]

2543. (Proposed by M. W. CROFTON, B.A.)—1. Given any closed convex boundary of length L, and another of length l lying wholly *inside* the former; the probability that a line drawn at random in the plane and meeting L shall also meet l, is l : L. (N.B.—If the boundaries are not convex, this expression still holds good, if L denote, not the length of the boundary, but that of a string tightly wrapped round it; likewise for l.)

2. If l be the length of a closed convex boundary lying wholly *outside* L, X the length of an endless band tightly enveloping the two boundaries and crossing between them, and Y the length of another endless band also enveloping both but not crossing between them, the probability that a random line meeting L shall also meet l, is $(X-Y)$: L.

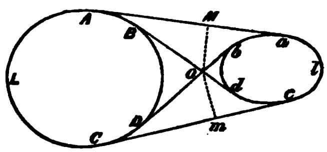
3. If the boundary l intersects L; then, if Y be the length of an endless band tightly enveloping both, the probability is $(L+l-Y)$: L.

Solution by CAPTAIN A. R. CLARKE, R.E., F.R.S.

1. Let O be any point within a closed convex curve, p, p' the perpendiculars on the tangents which make an angle θ with a given line; then $p+p$ is the breadth of the curve corresponding to the direction θ, and consequently $p+p'$ represents the number of random lines, parallel to θ, which cut the curve. Hence the total number of random lines cutting the curve is easily seen to be

$\int_0^\pi (p+p')\,d\theta$, which is, by a known theorem (Todhunter's *Int. Calc.*, p. 88),

equal to the perimeter of the curve. Hence, if there be a closed curve of length l within another of length L, the chance that a random line cutting the second also cuts the first is l : L.



2. Again, if l be without L; let $\theta=0$ correspond to Bd, θ_1 to Aa, θ_2 to Cc, θ_3 to Db. Let the perpendiculars on tangents be drawn from O, the intersection of the tangents Bd, Db. Let P', p' be perpendiculars on tangents to the upper parts of L and l respectively; $P_{,}$, $p_{,}$ perpendiculars on tangents at the lower parts, then a little consideration will show that the number of random lines crossing both curves is

$$(\text{between } 0 \text{ and } \theta_1) = \int_0^{\theta_1} (P'+p_{,})\,d\theta, \quad (\text{between } \theta_1 \text{ and } \theta_2) = \int_{\theta_1}^{\theta_2} (p'+p_{,})\,d\theta,$$

$$(\text{between } \theta_2 \text{ and } \theta_3) = \int_{\theta_2}^{\theta_3} (p'+P_{,})\,d\theta, \quad \text{which added are equal to}$$

$$\int_0^{\theta_1} P'\,d\theta\ (= \mathrm{AB}+\mathrm{BO}-\mathrm{AM}) + \int_{\theta_2}^{\theta_3} P_{,}\,d\theta\ (= \mathrm{CD}+\mathrm{DO}-\mathrm{C}m)$$

$$+ \int_{\theta_1}^{\theta_3} p'\,d\theta\ (= ab + \mathrm{O}b - a\mathrm{M}) + \int_0^{\theta_2} p_{,}\,d\theta\ (= cd + \mathrm{O}d - cm)$$

$$= \mathrm{AB}+\mathrm{CD}+ab+cd+\mathrm{D}b+\mathrm{B}d-\mathrm{A}a-\mathrm{C}c,$$

and this is easily seen to be $=$ X$-$Y, so that the probability in this case is (X$-$Y) : L.

3. If the curves intersect in two points, let the perpendiculars be dropped from a point O within both curves; then, if θ_1 be the inclination of the tangents Aa, Cc, the number of random lines crossing both curves is

$$\int_0^{\theta_1} (p_{,}+p')\,d\theta + \int_{\theta_1}^{\pi} (P'+p_{,})\,d\theta = \int_0^{\pi} p_{,}\,d\theta + \int_0^{\theta_1} p'\,d\theta + \int_{\theta_1}^{\pi} P'\,d\theta,$$

which, combining $p_{,}$ with p', and integrating the sum from 0 to $\pi+\theta$, is seen to be equal to $ca-a\mathrm{M}-cm+\mathrm{CA}-\mathrm{C}m-\mathrm{AM} = \mathrm{CA}+ca-\mathrm{C}c-\mathrm{A}a$, to which if we add and subtract ALC$+alc$, we shall find the required number to be $=$ L$+l-$Y. This expression will be found to hold good if the curves intersect in *four* points, in which case Y has four straight portions.

4. In a similar manner it may be shown that, if there be two non-intersecting convex closed curves, whose lengths are l, l, within a third whose

length is L, the chance that a random line intersecting L passes between l and l' is $(\text{X}-l-l') : \text{L}$.

NOTE.—If, as in the foregoing solution, we take a random line in accordance with the remarks of Mr. CROFTON on p. 85 of Vol. VII. of the *Reprint*, it may be similarly shown that, if two such random lines cross a closed convex curve whose length is L and area A, the chance that they intersect within the figure is $2\pi\text{A} : \text{L}^2$.

[Mr. CROFTON gives the following examples of the theorems, a straight line being viewed as an infinitely thin rectangle. A random straight line cuts a circle, the chance of its meeting an equal circle touching the first is $\frac{\pi-2}{\pi}$. The chance of a straight line, which cuts one side of a square, meeting the opposite side is $\sqrt{2}-1$; that of its meeting an adjacent side is $1-\frac{1}{2}\sqrt{2}$. The chance of a straight line, which meets a semicircle, meeting the opposite semicircle is $\frac{4}{\pi+2}$.]

2502. (Proposed by J. GRIFFITHS, M.A.)—If A, B, C, D be functions of x, y, z; and $\text{A}_1, \text{B}_1, \text{C}_1, \text{D}_1$ be certain constants; required the envelope of $\text{A}+\lambda\text{B}+\mu\text{C}+\nu\text{D}=0$, when the variables λ, μ, ν are connected by the two equations $\text{A}_1+\lambda\text{B}_1+\mu\text{C}_1+\nu\text{D}_1=0$, $\lambda^2+\mu^2+\nu^2-2\lambda\mu\nu=1$.

Solution by STEPHEN WATSON.

Eliminating first ν and then μ from the two first given equations, and substituting the resulting values of μ and ν in the third equation, we obtain

$$\text{K}\lambda^3+\text{L}\lambda^2+\text{M}\lambda+\text{N}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\text{K}=2(\text{BC}_1-\text{B}_1\text{C})(\text{DB}_1-\text{D}_1\text{B})$,

$$\text{L}=2(\text{AC}_1-\text{A}_1\text{C})(\text{DB}_1-\text{D}_1\text{B})+2(\text{BC}_1-\text{B}_1\text{C})(\text{DA}_1-\text{D}_1\text{A}) \\ -(\text{BC}_1-\text{B}_1\text{C})^2-(\text{DB}_1-\text{D}_1\text{B})^2-(\text{D}_1\text{C}-\text{DC}_1)^2,$$

$$\text{M}=2\left\{(\text{AC}_1-\text{A}_1\text{C})(\text{DA}_1-\text{D}_1\text{A})-(\text{AC}_1-\text{A}_1\text{C})(\text{BC}_1-\text{B}_1\text{C}) \\ -(\text{DA}_1-\text{D}_1\text{A})(\text{DB}_1-\text{D}_1\text{B})\right\},$$

$$\text{N}=(\text{D}_1\text{C}-\text{DC}_1)^2-(\text{AC}_1-\text{A}_1\text{C})^2-(\text{DA}_1-\text{D}_1\text{A})^2.$$

Differentiating (1) with respect to λ, we have

$$3\text{K}\lambda^2+2\text{L}\lambda+\text{M}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and eliminating λ from (1) and (2),

$$(\text{LM}-9\text{KN})^2=4(\text{L}^2-3\text{KM})(\text{M}^2-3\text{LN}) \quad\ldots\ldots\ldots\ldots (3);$$

whence, substituting the above values of K, L, M, N, we have the equation of the envelope required.

2415. (Proposed by Dr. SALMON, F.R.S.)—Find the equation of the evolute to the Cissoid $x(x^2+y^2)=ay^2$.

I. *Solution by* STEPHEN WATSON.

Since $y = \dfrac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}$, $\therefore\ \dfrac{dy}{dx} = p' = \dfrac{x^{\frac{1}{2}}(3a-2x)}{2(a-x)^{\frac{3}{2}}}$, and $\dfrac{d^2y}{dx^2} = p'' = \dfrac{3a^2}{4x^{\frac{1}{2}}(a-x)^{\frac{5}{2}}}$; hence, (α, β) being a point in the required locus, we have

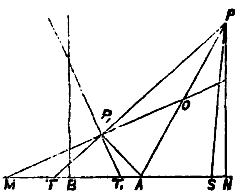
$$\alpha = x - \frac{p'(p'^2+1)}{p''} = \frac{ax(5x-6a)}{6(a-x)^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\beta = y + \frac{p'^2+1}{p''} = \frac{4ax^{\frac{1}{2}}}{3(a-x)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The elimination of x from (1) and (2) gives the required equation, which is, writing now (x, y) for (α, β),

$$27y^4 + 288a^2y^2 + 512a^3x = 0.$$

II. *Solution by* JAMES DALE.



Let PT be the tangent at any point (h, k) of a parabola whose vertex is A, focus S, and parameter $4a$; then P_1, the foot of a perpendicular from A on PT, is a point on the cissoid

$$x(x^2+y^2) + ay^2 = 0.$$

If O be the middle point of AP, then OP_1 is a normal to the locus of P_1; that is, the evolute to the cissoid is the envelope of OP_1.

The equation to OP_1 is

$$(y - \tfrac{1}{2}k) = (x - \tfrac{1}{2}h)\tan \text{OMA};$$

but $\angle\text{OMA} = \text{APS} = \text{APN} - \text{SPN}$, therefore $\tan\text{OMA} = \dfrac{ak}{h^2+k^2-ah}$.

Eliminating h by means of the equation $k^2 = 4ah$, the equation to OP_1 becomes $(12a^2k + k^3)y = 16a^3x + 4a^2k^2 + \tfrac{1}{4}k^4$.

Differentiating this with respect to k, and putting the result $= 0$, we get $2k = 3y$; and substituting in the equation to OP_1, and changing the sign of x, the equation to the evolute required is

$$y^2(9a^2 + \tfrac{27}{32}y^2) + 16a^3x = 0, \text{ or } 27y^4 + 288a^2y^2 + 512a^3x = 0.$$

[A solution by Mr. TUCKER is given on p. 58 of Vol. I. of the *Reprint*, where the form of the curve is shown. Mr. Tucker's result agrees with that given above, if we bear in mind that his a is the *radius* of the circular generator of the cissoid, whereas in the above solutions a is the *diameter* of the circle.]

2569. (Proposed by S. ROBERTS, M.A.)—If through a point O there be drawn three lines parallel respectively to the sides of a given triangle, the six points in which these lines meet the sides lie on a conic. Find its equation referred to the triangle. (See Question 2489.)

I. *Solution by* STEPHEN WATSON.

Take the figure in the Solution to Quest. 2489 in the January number, use triangular coordinates, and denote O by (x', y', z'); then the coordinates of I and F are $(0,\ 1-z',\ z')$ and $(0,\ y',\ 1-y')$; hence the equations of AI and AF are
$$y = \frac{1-z'}{z'}z' \quad\text{and}\quad \frac{1-y'}{y'}y = z,$$
and therefore by symmetry the equation of the conic through the six points is
$$\left(y - \frac{1-z'}{z'}z\right)\left(\frac{1-y'}{y'}y - z\right) + \left(z - \frac{1-x'}{x'}x\right)\left(\frac{1-z'}{z'}z - x\right) +$$
$$\left(x - \frac{1-y'}{y'}y\right)\left(\frac{1-x'}{x'}x - y\right) - \frac{1-x'}{x'}x^2 - \frac{1-y'}{y'}y^2 - \frac{1-z'}{z'}z^2 = 0, \text{ or}$$
$$\frac{1-x'}{x'}x^2 + \frac{1-y'}{y'}y^2 + \frac{1-z'}{z'}z^2$$
$$- \left(2 + \frac{z'}{x'y'}\right)xy - \left(2 + \frac{x'}{y'z'}\right)yz - \left(2 + \frac{y'}{z'x'}\right)zx = 0.$$

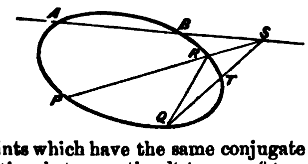
II. *Solution by the* PROPOSER.

Let (x, y, z) be the current trilinear coordinates referred to the given triangle. Then, denoting its sides by a, b, c, and writing L, M, N for $b\beta + c\gamma$, $a\alpha + c\gamma$, $a\alpha + b\beta$, where α, β, γ are the coordinates of the point O, we have for the coordinates of the six points of intersection
$$(0, \beta c, \mathrm{M}),\quad (\alpha b, \mathrm{L}, 0),\quad (\mathrm{N}, 0, \gamma a),$$
$$(0, \mathrm{N}, \gamma b),\quad (\mathrm{M}, \beta a, 0),\quad (\alpha c, 0, \mathrm{L});$$
and for conics through the first two pairs of points the equation is of the form
$$(\mathrm{LM}x - \alpha b\mathrm{M}y + \alpha\beta bcz)(\beta\gamma abx - \gamma b\mathrm{M}y + \mathrm{MN}z) + kxz = 0.$$
To determine k so that a conic of the system may pass through one of the remaining points, we have
$$k + \tfrac{1}{2}(\mathrm{L} + \mathrm{M} + \mathrm{N})(\mathrm{LMN} + abc\alpha\beta\gamma) = 0;$$
and substituting, we have
$$\frac{a\mathrm{L}}{\alpha}x^2 + \frac{b\mathrm{M}}{\beta}y^2 + \frac{c\mathrm{N}}{\gamma}z^2 - \left(\frac{\mathrm{LM}}{\alpha\beta} + ab\right)xy - \left(\frac{\mathrm{MN}}{\beta\gamma} + bc\right)yz - \left(\frac{\mathrm{LN}}{\alpha\gamma} + ac\right)zx = 0.$$
This may be put into a better form by adding $ax^2 + by^2 + cz^2$, and then subtracting the same. Thus we get
$$a(\beta\gamma x^2 - \alpha^2 yz) + b(\alpha\gamma y^2 - \beta^2 xz) + c(\alpha\beta z^2 - \gamma^2 xy) - \frac{\alpha\beta\gamma}{a\alpha + b\beta + c\gamma}(ax + by + cz)^2 = 0,$$
which is the required equation.

2554. (Proposed by Professor HIRST.)—Rays being drawn from any point on a conic; the segments intercepted thereon between the conic and a fixed

tangent can all be projected upon a given line, from another point on the conic, into segments of equal length.

I. *Solution by* W. S. McCay, B.A.; *the* Proposer; *and others.*



Draw any line through P meeting the conic and the line AB in R and S; and let Q be any other point on the conic. Then as the points R and T vary, RT divides the conic homographically, for clearly the anharmonic ratio of the four points R is that of the four points S, or that of the four points T. Also A and B are the two points which have the same conjugate considered as of either set. Now the relation between the distances of two conjugates of a homographic range on a line from a fixed point on the line is

$$axx' + bx + cx' + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and the equation that gives the points that have the same conjugates in either system is $\quad ax^2 + (b+c)\,x + d = 0$;

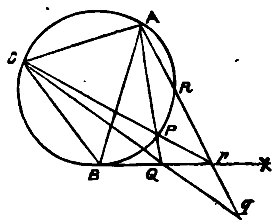
and if these are both infinite, then $a = 0,\ b + c = 0 \ldots\ldots\ldots\ldots\ldots (2)$,

and (1) reduces to $x - x' = -\dfrac{d}{b}$, that is, the distance between two conjugates is constant.

Now (2) will be the case, in the above figure, for a transversal across the homographic pencil whose vertex is Q, provided the line AB touch the conic, and the transversal be parallel to the line joining Q to the point of contact.

Hence the required construction for Q:—Draw a line through the point of contact of the tangent parallel to the direction of the given line: it meets the conic in Q.

II. *Solutions by* C. W. Merrifield, F.R.S.; H. M. Taylor, B.A.; J. J. Walker, M.A.; *and others.*



1. The theorem being a perspective one, the conic may be made a circle, and the given line may be drawn through the given point, without loss of generality.

Let ARBC be the given circle, BX the given tangent, ARpq the given line, and APQ a variable transversal.

Draw BC through the point of contact B, parallel to the given line, and cutting the conic in C; then C is evidently the point of sight required. For the segment of AB is infinitesimal, and its projection on AX can only be finite when the point of sight is on BC. As it is to be on the conic, it must be at B or C. It is evidently not at B, for the projection from B varies from zero to RX as Q moves from B to X. Hence it can only be at C, and it remains simply to show that the length pq in the figure is independent of the direction AQ.

Take BX, BC for Cartesian axes, and put $\angle\,\text{CBX} = \omega$, $\text{BC} = y_2$, $\text{BQ} = x$; then the equation to the circle is

$$U \equiv x^2 + y^2 + 2xy\cos\omega - yy_2 = 0.$$

Let A be (x_0, y_0): the equations to AQ and BC will be $y = y_0 \dfrac{x-x_1}{x_0-x_1}$ and $x=0$, and the quadratic equation of the pair of lines may be written as

$$V \equiv xy - \frac{y_0 x^2}{x_0-x_1} + \frac{x_1 y_0 x}{x_0-x_1} = 0.$$

Again, BA is $y - \dfrac{y_0}{x_0} x = 0$; and if we write CP as $\lambda x = y - y_2$, where λ is indeterminate, we have

$$BA.CP \equiv V_1 \equiv y^2 - \left(\lambda + \frac{y_0}{x_0}\right) xy + \frac{\lambda y_0}{x_0} x^2 - yy^2 + \frac{y_0 y_2}{x_0} x = 0.$$

Now U, V, V_1 have a common intersection in four points, and therefore we may determine λ by writing $\mu V + V_1 = \nu U$; and, equating coefficients, this

gives $\qquad \nu = 1, \quad \dfrac{y_2}{x_0} + \dfrac{\mu x_1}{x_0-x_1} = 0, \quad \mu - \lambda - \dfrac{y_0}{x_0} = 2\cos\omega;$

whence $\qquad \lambda = \dfrac{x_1-x_0}{x_0 x_1} y_2 - \dfrac{y_0}{x_0} = 2\cos\omega.$

There is a third indeterminate equation $\dfrac{\lambda y_0}{x_0} = \dfrac{\mu y_0}{x_0-x_1} + 1$, which, combined with the others, leads back to the equation of the circle. The y coordinate of p is therefore $\qquad y_2 + \lambda x_0 = 2y_2 - \dfrac{x_0}{x_1} y_2 - y_0 - 2x_0 \cos\omega,$

and the y coordinate of q is $y_2 - \dfrac{x_0}{x_1} y_2$, while their x coordinates are the same. The distance between them is therefore $pq = y_2 - y_0 - 2x_0\cos\omega$, and it is thus independent of the direction of AQ. The theorem is therefore proved.

2. The theorem is important as well as interesting, and it therefore seemed desirable to give an elementary proof of it. But the following trilinear proof by Mr. H. M. TAYLOR is both shorter and more general.

The point C being found as in Mr. MERRIFIELD'S Solution, let ABC be the triangle of reference; then the equation to the circumscribed conic is

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the tangent BX is $\dfrac{\alpha}{l} + \dfrac{\gamma}{n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

The equations to AR, AQ may be written $a\alpha = 2\Delta$, $\beta = k\gamma$ ...... (3, 4), where k is an arbitrary constant.

From (1) and (4) we obtain as the equation to CP

$$l\beta + (m + nk)\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Similarly, from (2) and (4), the equation to CQ is found to be

$$\frac{\alpha}{l} \mp \frac{\beta}{kn} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Let β_1, β_2 be the coordinates of p and q respectively; then

from (3) and (5), $$\beta_1 = -\frac{m+nk}{l}\cdot\frac{2\Delta}{a},$$

and from (3) and (6), $$\beta_2 = -\frac{nk}{l}\cdot\frac{2\Delta}{a};$$

therefore $$\beta_2 - \beta_1 = \frac{2m\Delta}{al} = \text{a constant};$$

therefore pq is also constant.

2464. (Proposed by J. J. WALKER, M.A.)—From a point in the plane of a conic circumscribing a triangle let lines be drawn to meet its three sides parallel respectively to the diameters bisecting those sides. If the area of the triangle formed by lines joining the feet of these parallels be constant, the locus of the point from which they are drawn will be a similar and coaxial conic.

Solution by MORGAN JENKINS, B.A.

Let T be any point in the locus; and let $T\alpha$, $T\beta$, $T\gamma$, drawn in the given directions, meet the sides BC, CA, AB respectively in α, β, γ.

Then $$\Delta\alpha\beta\gamma = \Delta T\beta\gamma + \Delta T\gamma\alpha + \Delta T\alpha\beta = \text{a constant};$$

therefore $lyz + mzx + nxy = $ a constant is the trilinear equation to the locus, l, m, n being constants depending only on the given directions in which $T\alpha$, $T\beta$, $T\gamma$ are drawn.

But by Quest. 2395 (the generalization of a well-known theorem) straight lines drawn through any point on the given conic in the same directions meet the sides of the triangle ABC in collinear points; therefore $lyz + mzx + nxy = 0$ is the equation of the given conic, and therefore the required locus is a conic meeting the given conic only at infinity; whence follows the theorem enunciated above.

[Orthogonal projection from the circle proves the theorems in Questions 2395 and 2464 for the ellipse, after which there cannot be much doubt of their general truth.]

2493. (Proposed by Professor CAYLEY.)—1. Given the conic $U = 0$ (but observe that the *function* U contains implicitly an arbitrary constant factor which is *not* given) and also the conic $U + 1 = 0$, to construct the conic $U + l = 0$, where l is a given constant.

2. Given the conics $U = 0$, $U + 1 = 0$, $V = 0$, $V + 1 = 0$, and the constants θ, k, to construct the conic $\theta U + \theta^{-1} V + 2k = 0$.

Solution by the PROPOSER.

1. The conics $U = 0$, $U + 1 = 0$, $U + l = 0$ are obviously concentric similar and similarly situated conics, and if drawing a line in any direction from the

centre, the radius-vectors for the three conics respectively are r, r', R, then it is easy to see that we have

$$R^2 = lr'^2 + (1-l)r^2.$$

There is no difficulty in constructing geometrically the radius R, and then the conic $U + l = 0$ is given as the concentric similar and similarly situated conic passing through the extremity of this radius.

2. To construct the conic $\theta U + \theta^{-1} V + 2k = 0$. By what precedes, we may construct the two conics $\theta U + k = 0$, $\theta^{-1} V + k = 0$; the four points of intersection of these lie on the required conic $\theta U + \theta^{-1} V + 2k = 0$, and also on the conic $\theta U - \theta^{-1} V = 0$; which last conic is consequently given as a conic passing through the four points in question, and also through the four points of intersection of the given conics $U = 0$, $V = 0$. But the conic $\theta U - \theta^{-1} V = 0$ being constructed, the conic $\theta U + \theta^{-1} V = 0$ can also be constructed; viz., the tangents of these two conics and of the conics $U = 0$, $V = 0$, at each of the four intersections $U = 0$, $V = 0$, form a harmonic pencil; and we have thus the conic $\theta U + \theta^{-1} V = 0$ a conic passing through four given points, and having at each of these a given tangent. And then finally the required conic $\theta U + \theta^{-1} V + 2k = 0$ is given as a conic concentric similar and similarly situated with the conic $\theta U + \theta^{-1} V = 0$, and passing through the four given points

$$\theta U + k = 0, \quad \theta^{-1} V + k = 0.$$

3. Treating k as an absolute constant but θ as a variable parameter, the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, is the quartic curve $UV - k^2 = 0$. This is a curve used by Plücker (in the *Theorie der algebraischen Curven*) for the purpose of showing that the 28 double tangents of a quartic curve may be all of them real. In fact, if $U = 0$, $V = 0$, be ellipses intersecting in four real points; and if, moreover, the implicit constants be such that U is positive for points *without* the first ellipse, V positive for points *within* the second ellipse, then since UV, $= k^2$, is positive for all points of the curve in question, the curve must be wholly situate in the four closed spaces which lie outside the one and inside the other of the two ellipses; consisting therefore of four detached portions. And when k is sufficiently small, then the figure of each portion is that of a concavo-convex lens with its angles rounded off: viz., each such portion has a real double tangent of its own. Any two portions have obviously four real double tangents, and hence the total number of real double tangents is $4 + 6 \times 4$, $= 28$.

4. A construction has been given by Aronhold (*Berl. Monatsber.*, July, 1864) by which, taking any 7 given lines as double tangents of a quartic curve, the remaining 21 double tangents can be constructed, and which, when the seven given lines are real, leads to a system of 28 real double tangents; but wishing to construct the figure of the 28 real double tangents, it occurred to me that the easier manner might be to construct Plücker's curve $UV - k^2 = 0$, as the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, and then to draw the tangents of this curve: the construction is, however, practically one of considerable difficulty, and I have not yet accomplished it.

2593. (Proposed by R. MOON, M.A.)—A cylindrical tube, closed at its lower end, and having an air-tight piston capable of moving freely in the upper part of it, rests vertically upon a solid pier. Above the piston there is a vacuum: below it there is air which at the time t is destitute of velocity, but of which the density varies as ϵ^x; where x is the distance of a stratum

from the base of the cylinder. A constant pressure equal to the pressure on the piston of the air below it at the time t is exerted upon the piston downwards. Explain the subsequent motion, the air and piston being supposed without weight.

Solution by the PROPOSER.

If the received law of pressure, viz., $p = a^2\rho$, holds in the above case, since the pressure of the base of the tube on the air above it must be exactly the same as that which the air above it exerts upon it, each particle of the air at the time t will have a uniform effective force f acting upon it; so that at $t+dt$ each particle will have acquired a velocity fdt, and will have described a space $\frac{1}{2}fdt^2$, at the same time that the density throughout will be exactly the same as before. Hence, at the time $t+2dt$ the velocity acquired will be $2fdt$, and the space described will be $\frac{1}{2}f(2dt)^2$; and so for each successive interval; so that, at the time $t+t_1$ the air will have acquired a velocity ft_1, and will have penetrated the pier to a depth $= \frac{1}{2}ft_1^2$; which is absurd.

It is clear that in the above case $\dfrac{dp}{dx}$ must vanish at the base of the cylinder, and increase in value according to some law as we recede from the base.

2483. (Proposed by C. M. INGLEBY, LL.D.)—What is the condition of the possibility of a square number being separable into two constituent squares, in more ways than one?

Solution by the PROPOSER.

If a given square number be separable into two constituent squares, it must be of the form $\left(\dfrac{p^2+q^2}{pq}\right)^2 r^2$, and then its constituent squares are

$$\left(\frac{p^2-q^2}{pq}\right)^2 r^2 \quad \text{and} \quad 4r^2.$$

[This has been (in effect) proved by the Solution of Question 1974, on p. 39 of Vol. VI. of the *Reprint*.]

If the given square number be separable thus in n ways, we must have

$$\left(\frac{p_1^2+q_1^2}{p_1q_1}\right)^2 r_1^2 = \left(\frac{p_2^2+q_2^2}{p_2q_2}\right)^2 r_2^2 = \dots\dots = \left(\frac{p_n^2+q_n^2}{p_nq_n}\right)^2 r_n^2,$$

or,
$$\frac{p_1^2+q_1^2}{p_1q_1} r_1 = \frac{p_2^2+q_2^2}{p_2q_2} r_2 = \dots\dots = \frac{p_n^2+q_n^2}{p_nq_n} r_n.$$

Thus, for instance, if we desire to divide 65^2 into two square numbers, we have $p_1=7$, $q_1=4$, and $r_1=1$, which gives $65^2 = 33^2+56^2$; and also $p_2=3$, $q_2=2$, and $r_2=5$, which gives $65^2 = 25^2+60^2$. All other cases of the separation of 65^2 are imaginary.

2562. (Proposed by A. W. PANTON, B.A.)—Show that the equation of a Cartesian oval, referred to the triple focus as origin, is

$$\left\{x^2+y^2-(\beta\gamma+\gamma\alpha+\alpha\beta)\right\}^2+4\alpha\beta\gamma\left\{2x-(\alpha+\beta+\gamma)\right\}=0,$$

where α, β, γ are the distances of the three single foci from the triple focus.

Solution by W. S. McCAY, B.A.

It is shown in Salmon's *Higher Plane Curves*, that the equation of a Cartesian oval may be written $S^2=b^3L$, where S is a circle and b a line; and by taking the origin at the centre of S (the triple focus) and the axis of y parallel to L, we may write it $(x^2+y^2-r^2)^2=b^3(x-a)$. Now if the line $x+iy=c$ touch [i being, as usual, put for $\sqrt{(-1)}$], the real point (if any) on it is a focus. Substitute for x, then

$$(c^2-2icy-r^2)^2=b^3(c-a-iy).$$

The condition for equal roots in y, arranged in powers of c, is

$$c^3-2c^2a+r^2c+\tfrac{1}{8}b^3=0.$$

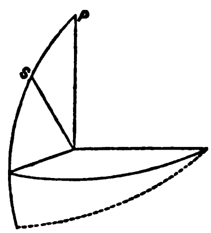
The values of c are the distances from the origin of the foci on the axis of x, the i having disappeared;

therefore $\quad 2c^2=\alpha+\beta+\gamma, \quad r^2=\beta\gamma+\gamma\alpha+\alpha\beta, \quad \tfrac{1}{8}b^3=-\alpha\beta\gamma.$

Putting in these values, we have the equation given in the question.

2421. (Proposed by M. W. CROFTON, B.A.)—Two places are taken at random in the Northern Hemisphere; find the chance of their distance exceeding 90° of a great circle.

Solution by the REV. J. WOLSTENHOLME, M.A.



One point S may be assumed on a given meridian, at a distance θ from the pole. Then, drawing the great circle of which S is pole, we see that the other point will be within 90° of S, unless it lie on a certain line whose area is to that of the hemisphere as $\theta:\pi$; and the chance of S lying between θ and $\theta+d\theta$ is $\sin\theta\, d\theta$. The chance, then, that the two points shall *not* lie within 90° of each other

is $$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\theta\sin\theta\, d\theta\equiv\frac{1}{\pi}.$$

2536. (Proposed by C. W. MERRIFIELD, F.R.S.)—When x becomes very large, the number of primes between $x-a$ and $x+a$ approaches the limit $\dfrac{2a}{\gamma+\log x}$, where $\gamma=0\,57721....$ (Euler's constant.)

Solution by the PROPOSER.

I refer to Euler (*Introductio in Analysin Infinitorum*, Cap. xv.) for the following theorem: If P be the sum of the series of the reciprocals of the prime numbers, and S the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$; then $P=\log S+C$, where C is a finite constant. By an unverified calculation, I find

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots\text{(primes)} = \log\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\text{(natural nos.)}\right\} + 0{\cdot}68428.$$

Moreover the sum of the harmonic series is $=\log x+0{\cdot}57721\ldots+\frac{1}{2x}-\ldots$
Now, if x be taken large, the *frequency* of the primes between definite limits x and $x+h$ is measured by the ratio between ΔS_x and ΔP_x, the limiting ratio of which is

$$\frac{d}{dx}S_x : \frac{d}{dx}P_x \text{ or } \frac{d}{dx}(\gamma+\log x) : \frac{d}{dx}\left\{\log(\gamma+\log x)+c\right\} \text{ or } \frac{1}{x} : \frac{1}{x(\gamma+\log x)},$$

whence the theorem.

Ex.: To find the number of primes between 3,034,000 and 3,036,000.

$$\text{Log } 3035000 = 14{\cdot}92572,$$

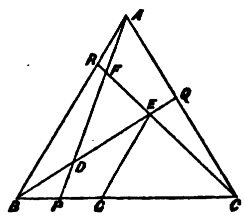
$$\gamma+\log x = 15{\cdot}503 \text{ and } \frac{2000}{15{\cdot}503} = 129.$$

The number counted is 128.

It will be observed that this method does not determine the constant B in the well-known form, when x is large, no. of primes under $x = \frac{x}{A\log x+B}$. It suggests that A should be found = unity, but not why $B=-1{\cdot}08366$.

2601. (Proposed by Capt. A. R. CLARKE, R.E., F.R.S.)—Points P, Q, R, are taken at random on the sides of a triangle ABC; show that the average area of the triangle formed by the intersections of AP, BQ, CR, is $10-\pi^2$.

Solution by S. WATSON; *the* PROPOSER; *and others.*



Let DEF be the triangle, and draw EG parallel to AB. Put $BR=cx$, $CQ=by$; then we easily find

$$EG=\frac{cxy}{x+y(1-x)},$$

$$\therefore \frac{\Delta BEC}{\Delta BAC}=\frac{EG}{AB}=\frac{xy}{x+y(1-x)} \ldots\ldots (1).$$

Hence the average area of the triangle BEC, in parts of the area of the triangle BAC, is

$$\int_0^1\int_0^1 \frac{xy\,dx\,dy}{x+y(1-x)} = \int_0^1 \left\{\frac{x}{1-x} + \frac{x^2}{(1-x)^2}\log x\right\} dx$$

$$= \int_0^1 \left\{\frac{1-x}{x} + \frac{(1-x)^2}{x^2}\log(1-x)\right\} dx$$

$$= \left\{\frac{1-x^2}{x}\log(1-x) + 2x\right\}_1^0 + \int_1^0 \frac{2}{x}\log(1-x)\,dx$$

$$= -3 + \tfrac{1}{3}\pi^2.$$

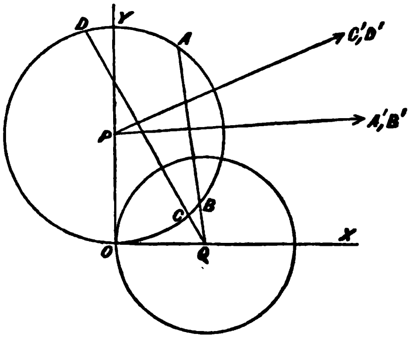
By symmetry the average area of each of the triangles CFA, ADB, is the same; hence the average area of DEF is $1 - 3(-3 + \frac{1}{3}\pi^2) = 10 - \pi^2$, [the same result as in Question 2537 and the third part of Question 2557].

2451. (Proposed by Professor CAYLEY.)—If A, B, C, D are the intersections of a conic by a circle, then the antipoints of A, B, and the antipoints of C, D, lie on a confocal conic.

N.B.—If AB, A B′ intersect at right angles in a point O in such wise that $OA' = OB' = i . OA = i . OB$ [where $i = \sqrt{(-1)}$ as usual], then A′, B′ are the anti-points of A, B, and conversely.

Solution by F. D. THOMSON, M.A.

1. The following construction will be found to give the antifocal points to the four points A, B, C, D on a circle (P):—



"Produce AB, CD to meet in Q, and with Q as centre describe the circle (Q), to cut the circle (P) orthogonally. The points where the perpendiculars from P on AB, CD meet the circle (Q) are the antifocal points required."

2. To show that any conic S round ABCD meets any conic S′ round A′B′C′D′ in four points which lie on a circle.

Take as axes the tangents to the two circles at one of their points of intersection. Then the equations are

$$(P) \equiv x^2 + y^2 - 2by = 0, \qquad (Q) \equiv x^2 + y^2 - 2ax = 0,$$

$$[AB] \equiv y - m(x-a) = 0, \qquad [A'B'] \equiv m(y-b) + x = 0,$$

$$[CD] \equiv y - m'(x-a) = 0, \qquad [C'D'] \equiv m'(y-b) + x = 0;$$

$$\therefore\ S \equiv [AB]\ [CD] + \kappa(P) \equiv \left\{y - m(x-a)\right\}\left\{y - m'(x-a)\right\} \\ + \kappa\left\{x^2+y^2-2by\right\} = 0,$$

and $$S' \equiv [A'B'][C'D'] + \lambda(Q) \equiv \left\{m(y-b)+x\right\}\left\{m'(y-b)+x\right\} \\ + \lambda\left\{x^2+y^2-2ax\right\} = 0;$$

or $$S \equiv y^2 - (m+m')xy + mm'x^2 + a(m+m')y - 2mm\,ax + mm'a^2 \\ + \kappa\left\{x^2+y^2-2by\right\} = 0,$$

and $$S' \equiv mm'y^2 + (m+m')xy + x^2 - 2bmm'y - b(m+m')x + mm'b^2 \\ + \lambda\left\{x^2+y^2-2ax\right\} = 0;$$

$$\therefore\ S+S' \equiv (1+mm')(x^2+y^2) + \left\{a(m+m') - 2bmm'\right\}y \\ - \left\{b(m+m') + 2amm'\right\}x + mm'(a^2+b^2) + (\kappa+\lambda)(x^2+y^2) - 2b\kappa y - 2a\lambda x = 0 \\ \equiv (1+mm'+\kappa+\lambda)(x^2+y^2) + \left\{a(m+m') - 2bmm' - 2b\kappa\right\}y \\ - \left\{b(m+m') + 2amm' + 2a\lambda\right\}x + mm'(a^2+b^2) = 0,$$

which is the equation to a circle passing through the four points of intersection of S and S′.

3. To find the conditions that S and S′ may be confocal.

If $S \equiv (A, B, C, F, G, H)(x, y, 1)^2 = 0$, so that $A = mm' + \kappa$, $B = 1 + \kappa$, $C = mm'a^2$, $F = \tfrac{1}{2}a(m+m') - b\kappa$, $G = -mm\,a$, $H = \tfrac{1}{2}(m+m')$; the general equation to a conic confocal with $S=0$ is (Salmon's *Conics*, Art. 384)

$$S + \frac{\mu}{\Delta}\Big[(AB-H^2)(x^2+y^2) - 2(HF-BG)x - 2(GH-AF)y \\ + BC - F^2 + CA - G^2\Big] + \frac{\mu^2}{\Delta} = 0.$$

Hence, if this coincide with $S' = 0$, that is, with

$$S - \Big[(1+mm'+\kappa+\lambda)(x^2+y^2) + \left\{a(m+m') - 2bmm' - 2b\kappa\right\}y - L\Big] = 0,$$

we must have

$$-\frac{\Delta}{\mu} = \frac{AB-H^2}{1+mm'+\kappa+\lambda} = \frac{-2(GH-AF)}{a(m+m') - 2bmm' - 2bx} = \frac{2(HF-BG)}{b(m+m') + 2amm' + 2a\lambda} \\ = \frac{BC - F^2 + CA - G^2 + \mu}{mm'(a^2+b^2)}.$$

Now, substituting the values of A, B, C, &c., and reducing, it will be found that these equations are *consistent*, and that they reduce to

$$-\frac{\Delta}{\mu} = \frac{(1+\kappa)(mm'+\kappa) - \frac{1}{4}(m+m')^2}{1+mm'+\kappa+\lambda} = k; \quad \text{or that} \quad \lambda = -\frac{(m-m')^2}{4\kappa},$$

$$\mu = -\frac{\Delta}{k} = \frac{a^2}{4}(m-m')^2 - \kappa\left[mm'(a^2-b^2) + ab(m+m')\right] + \kappa^2 b^2.$$

Substituting these values, we get a conic round A′B′C′D′ confocal with that round ABCD.

2594. (Proposed by C. T. HUDSON, LL.D.)—Let it be required to deal out ab cards n times into a sub-packs of b cards each, and so to arrange the sub-packs after each deal, that after the nth deal and arrangement of the sub-packs any selected card may stand in the rth place from the top of the whole pack.

Solution by the PROPOSER.

1. Suppose the selected card to stand after the first deal sth in its own pack, always counting as first the card which is undermost in any sub-pack, all the cards being dealt face upwards, and the sub-packs when re-arranged into one pack being held by the dealer so that the backs of the cards are towards him.

Let also $p_1, p_2, p_3 \dots\dots p_n$ be the places that the sub-pack of the selected card is to hold after the 1st, 2nd, 3rd .... nth regatherings of the sub-packs into one pack. Then after the 1st regathering the selected card will stand in the $b(p_1-1)+s$'th place from the top of the cards, and after the 2nd in the $b(p_2-1)+\dfrac{b(p_1-1)+s+m_1}{a}$'th place from the top of the cards, where m_1 lies between 0 and $a-1$, and must be so taken as to make $b(p_1-1)+s+m$ the least possible multiple of a. After the 3rd regathering the selected card will stand in the

$$b(p_3-1)+\frac{1}{a^2}\Big[b\left\{a(p_2-1)+(p_1-1)\right\}+s+m_1\Big]+\frac{m_2}{a}\text{'th place,}$$

$$\text{or in the } \frac{1}{a^2}\Big[b\left\{a^2(p_3-1)+a(p_2-1)+(p_1-1)\right\}+s+m_1+am_2\Big]\text{'th place,}$$

where m_1 and m_2 lie between 0 and $a-1$; and after the nth regathering in the

$$\frac{1}{a^{n-1}}\Big[b\left\{a^{n-1}(p_n-1)+a^{n-2}(p_{n-1}-1)+\dots+a^2(p_3-1)\right.$$
$$\left.+a(p_2-1)+(p_1-1)\right\}+s+m_1+am_2+a^2m_3+\dots+a^{n-2}m_{n-1}\Big]\text{'th.}$$

Now $\dfrac{1}{a^{n-1}}\left\{m_1+am_2+a^2m_3+\dots+a^{n-2}m_{n-1}\right\}$ lies between

$$0 \text{ and } \frac{1}{a^{n-1}}(a-1)(1+a+a^2+\dots+a^{n-2}), \quad \text{or between 0 and } \frac{a^{n-1}-1}{a^{n-1}},$$

and since a is a whole number and positive, this latter limit is always <1, and therefore the card's place, or r, lies between

$$\frac{1}{a^{n-1}}\Big[b\left\{(p_1-1)+a(p_2-1)+a^2(p_3-1)+\dots+a^{n-1}(p_n-1)\right\}+s\Big] \text{ and}$$
$$\frac{1}{a^{n-1}}\Big[b\left\{(p_1-1)+a(p_2-1)+a^2(p_3-1)+\dots+a^{n-1}(p_n-1)\right\}+s\Big]+1;$$

and therefore
$$p_1+ap_2+a^2p_3+\dots+a^{n-1}p_n \quad\dots\dots\dots\dots\dots\dots (\alpha)$$

lies between $\dfrac{1}{b}\left\{a^{n-1}r+b\dfrac{a^n-1}{a-1}-s\right\}$ and $\dfrac{1}{b}\left\{a^{n-1}(r-1)+b\dfrac{a^n-1}{a-1}-s\right\}$;

and since s may be as large as $b-1$ and as small as 1, therefore the expression (α) lies between

the nearest integer not greater than $\frac{1}{b}\left\{a^{n-1}r + b\frac{a^n-1}{a-1} - (b-1)\right\}$ ... (β),

and the nearest integer not less than $\frac{1}{b}\left\{a^{n-1}(r-1) + b\frac{a^n-1}{a-1} - 1\right\}$ ... (γ);

therefore, to determine $p_1, p_2, p_3 \dots p_n$, any integer lying between (β) and (γ) is to be divided n times by a, and the successive remainders will give, as is shown by (α), the quantities required. Should any quotient, or the original integer, be a multiple of a, then a itself must be taken as a remainder.

Ex. 1. Let there be 35 cards to be dealt 4 times into 5 sub-packs, and let it be required that the selected card should be in the 23rd place after the 4th deal. Here $a=5$, $b=7$, $n=4$, $r=33$; and substituting in (β) we find the upper limit to be 744; hence from

$$\left.\begin{array}{r|l} 5 & 744 \\ 5 & 148-4 \\ 5 & 29-3 \\ & 5-4 \end{array}\right\} \text{ we have } \left\{\begin{array}{l} p_1 = 4 \\ p_2 = 3 \\ p_3 = 4 \\ p_4 = 5 \end{array}\right.$$

2. The number of solutions is $\left\{(\beta) - p\right\} - \left\{(\gamma) + q\right\} + 1$, where p and q lie between 0 and $b-1$; and therefore when (β) and (γ) are both whole numbers, and consequently p and q are both zero, the number of solutions is $\frac{a^{n-1}+2}{b}$; but when (β) and (γ) are not whole numbers, the number of solutions is the nearest integer not less than $\frac{a^{n-1}+2}{b} - 1$ or than $\frac{a^{n-1}+2}{b} - 2$, according as $(p+q)$ lies between 0 and b, or b and $2b$. Of course, in any given case, the number of solutions would be obtained directly from (β) and (γ); but the above show that in no case is a solution possible if $a^{n-1}+2<b$.

3. By substituting ab, 1, $\frac{1}{2}(ab+1)$ in succession for r in (β) or (γ), it can be easily shown that, to bring a selected card to the bottom, top, or middle of the pack, after n deals, its sub-pack must always be placed at the bottom, top, or middle of the whole pack; and the smallest possible value of n is determined by the condition $a^{n-1}+2 =$ or $>b$.

The following is a simple case which leads to a neat and easily remembered formula by means of which many tricks may be performed with an ordinary pack of cards.

Let the number of cards be $a_1a_2a_3 \dots a_n$; and let the cards be dealt out into $a_1, a_2, a_3 \dots a_n$ sub-packs in succession. Then, with the same notation as before, the place of the selected card after the first deal will be the $a_2a_3a_4 \dots a_n(p_1-1)+s$ 'th, and after the second the

$$a_1a_3a_4 \dots a_n(p_2-1) + a_3a_4 \dots a_n(p_1-1) + \frac{s}{a_2} \text{ 'th};$$

or, since s lies between 1 and $a_2a_3a_4 \dots a_n$, the card's place lies between

$$\text{the } a_3a_4a_5 \dots a_n\left\{a_1(p_2-1)+(p_1-1)\right\}+1 \text{ 'th}$$

$$\text{and the } a_3a_4a_5 \dots a_n\left\{a_1(p_2-1)+(p_1-1)+1\right\} \text{ 'th}.$$

After the third deal it lies between

$$\text{the } a_4 a_5 \dots a_n \left\{a_1 a_2 (p_3-1) + a_1 (p_2-1) + (p_1-1)\right\} + 1 \text{ 'th}$$

$$\text{and the } a_4 a_5 \dots a_n \left\{a_1 a_2 (p_3-1) + a_1 (p_2-1) + (p_1-1) + 1\right\} \text{ 'th;}$$

and after the nth deal and regathering the two limits coincide; and then

$$r = 1 + (p_1-1) + a_1 (p_2-1) + a_1 a_2 (p_3-1) + \dots + a_1 a_2 a_3 \dots a_{n-1} (p_n-1),$$

and therefore $p_1 + a_1 p_2 + a_1 a_2 p_3 + \dots + a_1 a_2 a_3 \dots a_{n-1} p_n$

$$= r + a_1 + a_1 a_2 + a_1 a_2 a_3 + \dots + a_1 a_2 a_3 \dots a_{n-1} \dots (\delta);$$

and $p_1, p_2, p_3, \dots p_n$ may be found by dividing the right-hand member successively by $a_1, a_2, a_3 \dots a_n$; the remainders will be the quantities required.

If there are always at least 3 sub-packs and three deals, then the number of cards that may be used, by the above method, out of an ordinary pack of 52 cards, is $3 \times 3 \times 3$, $3 \times 3 \times 4$, $3 \times 3 \times 5$, or $3 \times 4 \times 4$, *i. e.*, 27, 36, 45, or 48.

In the first three cases, if the deals into 3 sub-packs are taken first, and those into 4 or 5 sub-packs last, the number to be added to r in (δ) will in each case be $3 + 3 \times 3$ or 12.

Ex. 2. Let there be 45 cards to be dealt out twice into 3 sub-packs of 15 each, and once into 5 sub-packs of 9 each, and let it be required to bring any selected card into the 41st place from the top after the third regathering of the cards.

Here $41 + 12 = 53$,

and from
$$\left.\begin{array}{r|l} 3 & 53 \\ 3 & 17-2 \\ & 5-2 \end{array}\right\} \text{ we have } \left\{\begin{array}{l} p_1 = 2 \\ p_2 = 2 \\ p_3 = 5 \end{array}\right.$$
an operation that can be readily performed mentally.

In the usual method given for doing this trick, the number of cards is always 27, and a long table of the proper values of p_1, p_2, p_3, necessary to produce the required result, must be committed to memory.

2434. (Proposed by Professor WHITWORTH.)—1. Three different persons have each to name an integer not greater than n. The chance that the integers named are proportional to the sides of some real triangle is

$$\tfrac{1}{2}\left(1 + \frac{1}{n^2}\right).$$

2. A person names a group of three integers (not necessarily different, but each one not greater than n). The chance that the integers named are proportional to the sides of some real triangle is

$$\tfrac{1}{2}\left\{1 + \frac{3(n+1)}{2n^2 + 4n + 1 + (-1)^n}\right\}.$$

Solution by Samuel Roberts, M.A.

Suppose that a, b, c are arranged according to magnitude, and all different. If these quantities are proportional to the sides of a real triangle, we have $c > a-b < b$, or $b > \frac{1}{2}a$. In the case of integers, we have the possible cases

$$\text{when } a \text{ is even}\left\{\begin{array}{r} a,\ \tfrac{1}{2}a+1,\ \tfrac{1}{2}a \\ \tfrac{1}{2}a+2,\ \tfrac{1}{2}a+1 \\ \tfrac{1}{2}a \\ \tfrac{1}{2}a-1 \\ \tfrac{1}{2}a+3,\ \tfrac{1}{2}a+2 \\ \tfrac{1}{2}a+1 \\ \tfrac{1}{2}a \\ \tfrac{1}{2}a-1 \\ \tfrac{1}{2}a-2 \\ \text{\&c., \&c.} \end{array}\right. \qquad \text{when } a \text{ is odd}\left\{\begin{array}{r} a,\ \tfrac{1}{2}(a+3),\ \tfrac{1}{2}(a+1) \\ \tfrac{1}{2}(a-1) \\ \tfrac{1}{2}(a+5),\ \tfrac{1}{2}(a+3) \\ \tfrac{1}{2}(a+1) \\ \tfrac{1}{2}(a-1) \\ \tfrac{1}{2}(a-3) \\ \text{\&c., \&c.} \end{array}\right.$$

The number of such combinations, when a is even, is $1+3+5+\ldots+a-3$ or $\frac{1}{4}(a-2)^2$; and when a is odd, the number is $2+4+6+\ldots+a-3$, or $\frac{1}{4}(a-1)(a-3)$. In the present question the limit of a is n. Suppose n even, and we have

$$2\left\{1^2+2^2+\ldots+\tfrac{1}{4}(n-2)^2\right\}-\left\{1+2+\ldots+\tfrac{1}{2}(n-2)\right\},$$

or $$\tfrac{1}{24}(2n^3-9n^2+10n)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\text{A}).$$

In like manner, when n is odd, we have $\frac{1}{24}(2n^3-9n^2+10n-3)$ ...... (A′).

Similarly, for combinations of the form (a, a, b), we get for n even

$\frac{1}{4}(3n^2-4n)$ .... (B), and for n odd $\frac{1}{4}(3n^2-4n+1)$ .... (B′);

and the number of cases of (a, a, a) is n in both cases.

As to the *first* part of the question, the total number of combinations is n^3; and we have for the required probability

$$\frac{2.3(\text{A})+3(\text{B})+n}{n^3}\ (n \text{ even}),\quad \text{and}\quad \frac{2.3(\text{A}')+3(\text{B}')+n}{n^3}\ (n \text{ odd}),$$

which are both comprehended under the form given.

As to the *second* part of the question, whether n is even or odd, the total number of cases is $\frac{1}{6}\left\{n(n+1)(n+2)\right\}$, and the required probability is

$$\frac{(\text{A})+(\text{B})+n}{\frac{1}{6}n(n+1)(n+2)}\ (n \text{ even}),\quad \text{and}\quad \frac{(\text{A}')+(\text{B}')+n}{\frac{1}{6}n(n+1)(n+2)}\ (n \text{ odd});$$

which are both comprehended under the form given.

[Prof. Whitworth obtains the following corollary, by supposing $n = \infty$. If three numbers be named at random, they are *as likely as not* to be proportional to the sides of some triangle.]

2558. (Proposed by C. W. Merrifield, F.R.S.)—Mr. Henry Goodwin published in 1818 a table, in which all proper fractions (reduced to their lowest terms) in which the denominator did not exceed 100, nor the nume-

rator 50, were arranged in order of magnitude. He observed the following property, a *general* proof of which is required. Let any three consecutive fractions be $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3}$, then $\frac{N_1+N_3}{D_1+D_3} = \frac{N_2}{D_2}$.

I. *Solution by the* PROPOSER.

The difference between two successive terms, $\frac{D_1N_2-D_2N_1}{D_1D_2}$, is always equal to $\frac{1}{D_1D_2}$; for if it were not, and if $D_1N_2-D_2N_1 = k$, where k is an integer greater than unity, we might solve $D_1N_2-D_2N_1 = 1$ as an indeterminate equation yielding an intermediate value, by means of which we could interpolate a term, which is contrary to hypothesis. Hence we have

$$\frac{N_2}{D_2}-\frac{N_1}{D_1}=\frac{1}{D_1D_2}, \text{ and } \frac{N_3}{D_3}-\frac{N_2}{D_2}=\frac{1}{D_2D_3}, \text{ whence } \frac{N_2}{D_2}=\frac{N_1+N_3}{D_1+D_3}.$$

II. *Solution by* MORGAN JENKINS, B.A.

To make the proof general, let D be the superior limit of the denominators, $\frac{N_m}{D_m}$ any one of the fractions expressed in its lowest terms, so that N_m is prime to D_m which is less than D. Let ν_m, δ_m be the least positive solutions of

$$D_m\nu - N_m\delta = +1 \text{ .... (1)};$$

then, if we take $N_{m+1} = \nu_m + tN_m$, $D_{m+1} = \delta_m + tD_m$ .... (2),

where $t = \mathrm{I}\left(\frac{D-\delta_m}{D_m}\right)$, *i. e.* the greatest integer in the fraction, $\frac{N_{m+1}}{D_{m+1}}$ will be the consecutive fraction of the series in ascending order of magnitude. For if $\frac{P}{Q}$ be any other fraction of the series, such that $PD_m - QN_m = r$ (positive), then

$$P = r\nu_m + sN_m, \quad Q = r\delta_m + sD_m.$$

Hence, if $r>1$, then since Q is $<D$, sD_m is $<D-r\delta_m < D-\delta_m < tD_m$;

and therefore $\quad s<t$ and $r\delta_m+sD < r(\delta_m+tD_m)$.

Also, if $r=1$, s is $<t$, as before, and δ_m+sD_m is $<\delta_m+tD_m$;

therefore in either case $\frac{N_{m+1}}{D_{m+1}}-\frac{N_m}{D_m}$ is $<\frac{P}{Q}-\frac{N_m}{D_m}$;

and this proves that $\frac{N_{m+1}}{D_{m+1}}$ is the next fraction in ascending order of magnitude. In like manner we shall have

$$N_{m-1} = \nu'_m + uN_m, \quad D_{m-1} = \delta'_m + uD_m \text{ .... (3)},$$

where $\nu'_m = N - \nu_m$, and $\delta' = D_m - \delta_m$, $u = I\left(\frac{D-\delta'_m}{D_m}\right)$;

therefore $N_{m+1} + N_{m-1} = vN_m$, $D_{m+1} + D_{m-1} = vD_m$ ....(4),

where $v = 1 + t + u = I\left(\frac{D + D_{m-1}}{D_m}\right) = I\left(\frac{D + D_{m+1}}{D_m}\right)$;

and this proves the theorem for a series of fractions between any given limits $\frac{N_1}{D_1}$ and $\frac{N_l}{D_l}$ (D_1 and D_l both less than D).

Equations (2) must be used to obtain the fraction next greater than $\frac{N_1}{D_1}$; or (3) to obtain the fraction next less than $\frac{N}{D_l}$; and then (4) to complete the series from any consecutive pair of fractions.

2323. (Proposed by S. Roberts, M.A.)—Determine the order, in the coefficients, of the conditions that three cubic equations may have a common root, the same being one of a pair of equal roots as to two of the given equations.

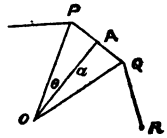
Solution by the Proposer.

There are four conditions represented by five equations of the degrees 2, 2, 2, 2, 3, in x, y, and of the same first order as to the coefficients. We have therefore to take $lmnpq \Sigma \frac{1}{lmnp}$ or Σl, substituting the above numbers for l, m, n, p, q. The result is 11. (Salmon's *Higher Algebra*, p. 281.)

2433. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Five points being taken at random on the surface of a regular polygon of n sides as the apices of a pentagon, show that the probability of two of them being re-entrant is

$$\frac{5}{12}\left\{\frac{\cos\frac{2\pi}{n} + 2}{n \sin\frac{2\pi}{n}}\right\}^2.$$

Solution by the Proposer.



Let PQR &c. be a portion of the boundary of the given polygon. By one of the general theorems stated in Question 2471,* if the axes of coordinates be principal axes drawn through the centre O, and h, k denote the radii of gyration of the polygon round those axes, then

$$(\Delta^2) = \frac{3}{2}\frac{h^2k^2}{M^2}, \text{ and } p = 10(\Delta^2) = \frac{15h^2k^2}{M^2},$$

* *Reprint*, Vol. VIII., p. 100.

where (Δ^2) is the average square of area of an arbitrary triangle drawn on the surface, expressed in parts of M^2, the square of the total area, and p the sought probability.

Now, as the figure is symmetrical, all rectangular axes drawn through the centre are principal axes, and $h=k$. Also, if l denote the radius of gyration round an axis through O perpendicular to the plane of the figure, then $l^2=h^2+k^2$. But if the radii OP, OQ, &c. be drawn, the polygon is composed of n equal triangles symmetrically placed round the centre, and therefore l is the same as the radius of gyration of any one of the triangles (POQ) about the same axis. Take OA bisecting PQ as the axis of x, and let $OA=a$, the angle $POA=\theta$, and the area of the triangle $POQ=m$; then $m=a^2\tan\theta$, and

$$m\,.\,l^2 = \Sigma\rho^2 dm = \iint dx\,dy\,(x^2+y^2) = \int dx\,(x^2y+\tfrac{1}{3}y^3)$$

(limits $y=-x\tan\theta\ldots+x\tan\theta$)

$$= 2\tan\theta\int x^3 dx\,(1+\tfrac{1}{3}\tan^2\theta) = \tfrac{1}{2}a^4\tan\theta\,(1+\tfrac{1}{3}\tan^2\theta);$$

therefore $l^2 = \frac{1}{2}a^2(1+\frac{1}{3}\tan^2\theta)$, and $h^2=k^2=\frac{1}{4}a^2(1+\frac{1}{3}\tan^2\theta)$.

Hence, since $M=n\,.\,m=na^2\tan\theta$, we obtain, by the foregoing formula,

$$(\Delta^2)=\frac{3}{32}\left(\frac{1+\frac{1}{3}\tan^2\theta}{n\tan\theta}\right)^2=\frac{1}{24}\left(\frac{3+\tan^2\theta}{2n\tan\theta}\right)^2=\frac{1}{24}\left(\frac{\cos2\theta+2}{n\sin2\theta}\right)^2,$$

$$p=10(\Delta^2)=\frac{5}{12}\left(\frac{\cos2\theta+2}{n\sin2\theta}\right)^2=\frac{5}{12}\left\{\frac{\cos\frac{2\pi}{n}+2}{n\sin\frac{2\pi}{n}}\right\}^2.$$

As examples, take $n=3, 4, 6, \infty$, and we obtain

for the triangle, $p=\dfrac{5}{12}\left(\dfrac{1\frac{1}{2}}{\frac{3}{2}\sqrt{3}}\right)^2=\dfrac{5}{36}$;

for the square, $p=\dfrac{5}{12}\left(\dfrac{2}{4}\right)^2=\dfrac{5}{48}$;

for the hexagon, $p=\dfrac{5}{12}\left(\dfrac{2\frac{1}{2}}{3\sqrt{3}}\right)^2=\dfrac{125}{1296}$;

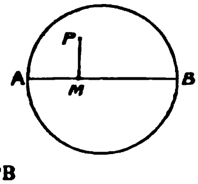
for the circle, $p=\dfrac{5}{12}\left(\dfrac{3}{2\pi}\right)^2=\dfrac{45}{48\pi^2}$.

2500. (Proposed by T. SAVAGE, M.A.)—1. If two points be taken at random within a circle, the chance that the line joining them will meet a given diameter at some point within the circle is $\dfrac{3\pi^2+16}{6\pi^2}$.

2. If three points be taken at random within a circle, the chance that these three points and the centre of the circle will form the angles of a re-entrant quadrilateral is $\dfrac{3\pi^2+16}{12\pi^2}$.

Solution by the REV. J. WOLSTENHOLME, M.A.; S. WATSON; *and others.*

1. Let AB ($=2a$) be the given diameter, P and Q the two points; then PQ will intersect AB in all cases when P and Q lie on opposite sides of AB. The chance of this is $\frac{1}{2}$. Now let x be the chance that PQ intersects AB when P and Q are on the same side; the whole chance will then be $\frac{1}{2}(1+x)$. But PQ will intersect AB if P lie within the triangle AQB, or Q within the triangle APB;



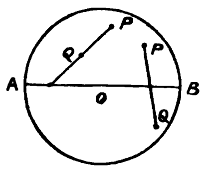
therefore $$x=\frac{2\times\text{average value of the }\triangle\text{APB}}{\text{semicircle}}$$

$$=\frac{\text{AB}\times\text{average value of the ordinate PM}}{\text{semicircle}}$$

$$=\frac{2}{\pi a^2}\cdot 2a\cdot\frac{4a}{3\pi}=\frac{16}{3\pi^2};$$

therefore the whole chance required is $\dfrac{3\pi^2+16}{6\pi^2}$.

2. One of the three points may be considered fixed on the circumference without loss of generality; if we take this point at A, O the centre, P and Q the other two points; then, for a re-entrant quadrilateral, we must have either (α) P and Q on the same side of AB and PQ produced intersecting AO, or (β) P and Q on opposite sides of AB and PQ intersecting BO.

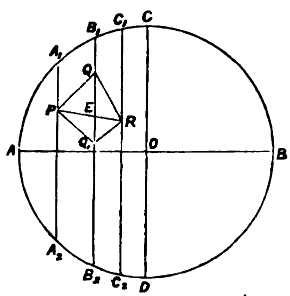


The chance of (α) is $\frac{1}{2}\cdot\frac{1}{2}\cdot x$, and of ($\beta$) is $\frac{1}{4}$; therefore the whole chance is $\frac{1}{4}(1+x)$ or $\dfrac{3\pi^2+16}{12\pi^2}$.

2422. (Proposed by the EDITOR.)—Find the average area of all the triangles that can be drawn within the circumference of a given circle, and deduce therefrom the probability that four points taken at random within the circle will form the apices of a convex quadrangle.

Solution by S. BILLS; T. SAVAGE, M.A.; *and others.*

In the accompanying diagram, let ACBD be a circle whose radius is unity, AB and CD being two diameters perpendicular to each other, intersecting in the centre O. Let PQR be one of the inscribed triangles, and through its vertices draw the lines A_1A_2, B_1B_2, C_1C_2, parallel to CD. It is obvious that we shall not destroy the generality of the question by taking the radius equal to unity, and it will greatly simplify the operations. Let Q_1 be the position of Q when below PR; and suppose B_1B_2 to intersect PR in E. Put $A_1P=x'$, $B_1Q=y'$, $C_1R=z'$; angle



$AQA_1 = \theta_1$, $AOB_1 = \theta_2$, $AOC_1 = \theta_3$; then $A_1A_2 = 2\sin\theta_1 = 2s_1$, $B_1B_2 = 2\sin\theta_2 = 2s_2$, $C_1C_2 = 2\sin\theta_3 = 2s_3$; also the perpendiculars from P, Q, R on CD will be, respectively, $\cos\theta_1 = c_1$, $\cos\theta_2 = c_2$, $\cos\theta_3 = c_3$. Now putting $B_1E = v$, we shall have

$$\frac{(s_1-x')-(s_3-z')}{(s_1-x')-(s_2-v')} = \frac{c_1-c_3}{c_1-c_2},$$

therefore $$v = \frac{(c_1-c_2)\,z' + (c_2-c_3)\,x' - (c_1-c_2)\,s_3 - (c_2-c_3)\,s_1}{c_1-c_3} + s_2.$$

Now when Q is *above* PR, we have area $PQR = \frac{1}{2}(c_1-c_3)(v-y')$,
and when Q_1 is *below* PR, we have area $PQ_1R = \frac{1}{2}(c_1-c_3)(y'-v)$.

Hence, $$\tfrac{1}{2}(c_1-c_3)\left\{\int_0^v (v-y')\,dy' + \int_v^{2s_2}(y'-v)\,dy'\right\}$$

$$= \tfrac{1}{2}(c_1-c_3)\left\{(v-s_2)^2+s_2^2\right\} \text{ .... } (1)$$

expresses the sum of the areas of PQR while Q ranges from B_1 to B_2.

Restoring the value of v as given above, we have

$$\int_0^{2s_1}\int_0^{2s_3}(1)\,dx'\,dz'$$

$$= \frac{\frac{2}{3}(c_1-c_2)^2 s_1s_3^3 + \frac{2}{3}(c_2-c_3)^2 s_1^3s_3 - 2(c_1-c_3)^2 s_1s_3c_2^2 + 2(c_1-c_3)^2 s_1s_3}{c_1-c_3} \text{ ...... } (2),$$

which expresses the sum of the areas of PQR, while P, Q, R range, respectively, over the lines A_1A_2, B_1B_2, C_1C_2. Now put

$$\tfrac{2}{3}s_1s_3^3 + \tfrac{2}{3}s_1^3s_3 - 2s_1s_3(c_1-c_3)^2 = L,\quad \tfrac{4}{3}c_1s_1s_3^3 + \tfrac{4}{3}c_3s_1^3s_3 = M,$$
$$\tfrac{2}{3}c_1^2s_1s_3^3 + \tfrac{2}{3}c_3^2s_1^3s_3 + 2s_1s_3(c_1-c_3)^2 = N;$$

then (2) will become $$\frac{Lc_2^2 - Mc_2 + N}{c_1-c_3} \text{ .... } (3).$$

Also $$\int_{c_3}^{c_1}(3)\,dc_2 = \frac{\frac{1}{3}L(c_1^3-c_3^3) - \frac{1}{2}M(c_1^2-c_3^2) + N(c_1-c_3)}{c_1-c_3}$$

$$= \tfrac{1}{3}L(c_1^2+c_1c_3+c_3^2) - \tfrac{1}{2}M(c_1+c_3) + N \text{ .... } (4)$$

will evidently express the sum of the areas of PQR while Q ranges over the entire band $A_1A_2C_2C_1A_1$. Now since the lines A_1A_2, B_1B_2, C_1C_2 may be interchanged in six different ways for the total sum (S suppose) of the areas, when P, Q, R range over the entire surface of the circle, will be

$$S = 6\int_0^\pi\int_0^{\theta_1}(4)\,.\,d(1-\cos\theta_3)\,.\,d(1-\cos\theta_1)$$

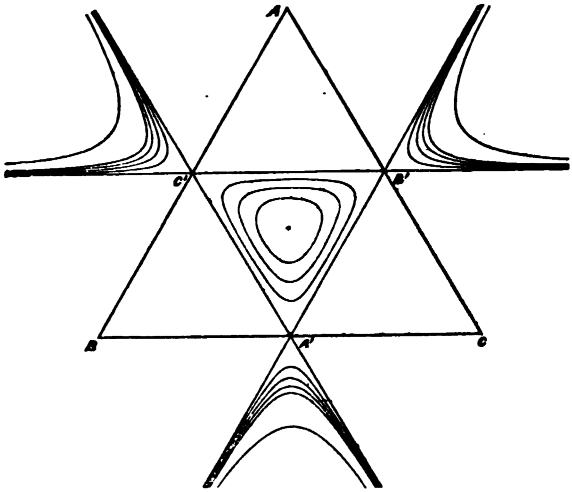
$$= 6\int_0^\pi\int_0^{\theta_1}(4)\,s_3s_1\,d\theta_3\,d\theta_1 \text{ .... } (5).$$

Restoring the values of L, M, N in (4) and substituting in (5); and then integrating by aid of the usual trigonometrical formula, we find $S = \frac{35}{48}\pi^2$.

Now the number of triangles may evidently be expressed by the cube (π^3) of the area of the circle, since each of the points P, Q, R may range over the entire surface. Hence the required average is $\frac{35\pi^2}{48} \div \pi^3 = \frac{35}{48\pi}$, for a circle of radius unity; and for a radius r it is $\frac{35r^2}{48\pi}$. The probability (p) of the inscribed quadrilateral being re-entrant, is therefore (see *Reprint*, Vol. VI., p. 52) four times this average divided by the area of the circle, or $p = \frac{35}{12\pi^2}$, which agrees with the result otherwise obtained on pp. 88 and 105 of Vol. VIII. of the *Reprint*. It is easy to see that the same result would be obtained for the ellipse.

2557. (Proposed by the EDITOR.)—Show (1) that the locus of the centre of an ellipse of constant area touching the three sides of a triangle is a cubic hyperbola; and, supposing the centres to be equably distributed over every possible position within the triangle, show (2) that the average area of all the inscribed ellipses, in parts of the area of the triangle, is $\frac{4}{105}\pi^2$; moreover, on the same supposition of equable distribution of centres, show (3) that the average area of the triangle formed by joining the points of contact of the inscribed ellipses, in parts of the area of the triangle, is $10-\pi^2$.

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.; *and* CAPT. CLARKE, F.R.S.



1. Taking two sides BC ($=a$) and BA ($=c$) of the triangle for axes of Cartesian coordinates (x and y), the equation of an inscribed conic is

$$\frac{x}{h}+\frac{y}{k}-1=2\lambda\left(\frac{xy}{hk}\right)^{\frac{1}{2}},$$

with the condition [that it should touch AC, viz.],

$$\frac{\lambda^2}{hk}=\left(\frac{1}{h}-\frac{1}{a}\right)\left(\frac{1}{k}-\frac{1}{c}\right)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The centre is immediately found to be $\frac{X}{h}=\frac{Y}{k}=\frac{1}{2(1-\lambda^2)}$;

and the equation of the conic referred to its centre is

$$\frac{x^2}{h^2}+\frac{2xy}{hk}(1-2\lambda^2)+\frac{y^2}{k^2}=\frac{\lambda^2}{1-\lambda^2}.$$

If this touch the circle $x^2+2xy\cos B+y^2=r^2$, we have

$$\left(\frac{r^2}{h^2}-\frac{\lambda^2}{1-\lambda^2}\right)\left(\frac{r^2}{k^2}-\frac{\lambda^2}{1-\lambda^2}\right)=\left(\frac{1-2\lambda^2}{hk}r^2-\frac{\lambda^2\cos B}{1-\lambda^2}\right)^2;$$

and the area (πn^2) is $\frac{\pi hk\lambda\sin B}{2(1-\lambda^2)^{\frac{3}{2}}}$,

whence $$n^2 = 2XY\lambda(1-\lambda^2)^{\frac{1}{2}}\sin B \;\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and by (1), $$\lambda^2 = \left\{1 - \frac{2X(1-\lambda^2)}{a}\right\}\left\{1 - \frac{2Y(1-\lambda^2)}{c}\right\},$$

or $$1 - \frac{2X}{a} - \frac{2Y}{c} + \frac{4XY}{ac}(1-\lambda^2) = 0 \;\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating λ from (1) and (3), we have

$$\left(1 - \frac{2X}{a} - \frac{2Y}{c} + \frac{2XY}{ac}\right)^2 = \frac{4X^2Y^2}{a^2c^2}\left(1 - \frac{n^4}{X^2Y^2\sin^2 B}\right),$$

or $$\left(1 - \frac{2X}{a} - \frac{2Y}{c}\right)\left(1 - \frac{2X}{a}\right)\left(1 - \frac{2Y}{c}\right) + \frac{4n^4}{a^2c^2\sin^2 B} = 0.$$

Now, if we refer to triangular coordinates (α, β, γ), taking as fundamental triangle A′B′C′, whose vertices are at the middle points of the sides of the original triangle (area $= \Delta$ suppose), we have

$$\alpha = \frac{2X}{a} + \frac{2Y}{c} - 1, \quad \beta = 1 - \frac{2X}{a}, \quad \gamma = 1 - \frac{2Y}{c};$$

and the equation of the required locus is $$\alpha\beta\gamma = \frac{4n^4}{a^2c^2\sin^2 B} = \frac{n^4}{\Delta^2} \;\ldots\ldots (4).$$

[The complete locus denoted by (4) is, for various values of n, a series of cubic hyperbolas lying between the asymptotes B′C′, C′A′, A′B′, as represented in the diagram. Of the outer hyperbolic branches, which are the loci of the centres of the escribed ellipses, the four inner individuals correspond respectively to the three closed interior curves and the isolated point the centre, this point being the position of the centre of the greatest inscribed ellipse.]

2. The average area, on the supposition of equable distribution of centres, of all inscribed ellipses lying within the triangle ABC, and whose centres therefore lie within A′B′C′, is, in parts of the area of ABC,

$$\pi\int_0^1\int_0^{1-\alpha}\left\{\alpha\beta(1-\alpha-\beta)\right\}^{\frac{1}{2}} d\alpha\, d\beta \div \int_0^1\int_0^{1-\alpha} d\alpha\, d\beta$$

$$= \frac{\pi^2}{4}\int_0^1 \alpha^{\frac{1}{2}}(1-\alpha)^2\, d\alpha = \frac{4\pi^2}{105}.$$

3. If the equation of the conic (referred to ABC) be

$$(p\alpha')^{\frac{1}{2}} + (q\beta')^{\frac{1}{2}} + (r\gamma')^{\frac{1}{2}} = 0,$$

the area of the triangle formed by joining the points of contact is, in parts of the triangle ABC,

$$1 - \frac{p^2}{(p+q)(p+r)} - \frac{q^2}{(q+r)(q+p)} - \frac{r^2}{(r+p)(r+q)} = \frac{2pqr}{(q+r)(r+p)(p+q)},$$

and the centre of this conic is $$\frac{\alpha'}{q+r} = \frac{\beta'}{r+p} = \frac{\gamma'}{p+q},$$

or, referring to triangle A′B′C′,

$$\frac{\beta+\gamma}{q+r} = \frac{\gamma+\alpha}{r+p} = \frac{\alpha+\beta}{p+q}, \text{ or } \frac{\alpha}{p} = \frac{\beta}{q} = \frac{\gamma}{r}.$$

Hence the area of the triangle joining the points of contact is $\dfrac{2\alpha\beta\gamma}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}$; and the average area of this triangle, on the supposition of equable distribution of centres, is

$$\int_0^1\int_0^{1-\alpha}\frac{2\alpha\beta\,(1-\alpha-\beta)\,d\alpha d\beta}{(1-\alpha)(1-\beta)(\alpha+\beta)}+\int_0^1\int_0^{1-\alpha}d\alpha d\beta$$

$$=4\int_0^1\frac{\alpha d\alpha}{1-\alpha}\int_0^{1-\alpha}\left\{1-\frac{\alpha}{1+\alpha}\left(\frac{1}{\alpha+\beta}+\frac{1}{1-\beta}\right)\right\}d\beta$$

$$=4\int_0^1\frac{\alpha d\alpha}{1-\alpha}\left\{1-\alpha+\frac{2\alpha}{1+\alpha}\log\alpha\right\}$$

$$=2+8\int_0^1(\alpha^2+\alpha^4+\ldots)\log\alpha d\alpha=2-8\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)$$

$$=2-8\left(\frac{\pi^2}{8}-1\right)=10-\pi^2.$$

II. *Solution by* STEPHEN WATSON.

1. Take BC ($=a$) and BA ($=c$) as axes, and let D, E, F be the points of contact of the inscribed ellipse with BC, CA, AB. Put BD $=a\alpha$, BF $=c\beta$; then the equation of the ellipse is

$$\left(\frac{y}{c\beta}-1\right)^2+\left(\frac{x}{a\alpha}-1\right)^2+2kxy-1=0,\text{ where }k=\frac{1-2\,(1-\alpha)(1-\beta)}{ac\alpha\beta}\quad(1)$$

and the coordinates of the centre are

$$l=\frac{a\alpha}{1+kac\alpha\beta}=\frac{a\alpha}{2\,(\alpha+\beta-\alpha\beta)},\quad m=\frac{c\beta}{2\,(\alpha+\beta-\alpha\beta)}\quad\ldots\ldots\ldots(2),$$

therefore $$\alpha=1+\frac{c\,(l-\frac{1}{2}a)}{am}\text{ and }\beta=1+\frac{a\,(m-\frac{1}{2}c)}{cl}\quad\ldots\ldots\ldots\ldots(3).$$

Now the area of the ellipse (1) is easily found to be

$$=\frac{\pi\alpha\beta ac\sin B}{(1+kac\alpha\beta)^2}\,(1+kac\alpha\beta)^{\frac{1}{2}}(1-kac\alpha\beta)^{\frac{1}{2}}\quad\ldots\ldots\ldots\ldots\ldots(4),$$

and putting this equal the constant $n\pi\sin B$, we easily find by the aid of (2) and (3), putting now (x, y) for (l, m), that the equation of the locus of the centre is

$$2\,(ay+cx-\tfrac{1}{2}ac)\,(\tfrac{1}{2}c-y)\,(\tfrac{1}{2}a-x)=n^2\quad\ldots\ldots\ldots\ldots\ldots(5),$$

[which is easily seen to be that of a cubic hyperbola, as represented in the diagram to the foregoing solution.]

2. From the above, the area of the ellipse in parts of the triangle is

$$\frac{2^{\frac{3}{2}}\pi}{ac}(ay+cx-\tfrac{1}{2}ac)^{\frac{1}{2}}(\tfrac{1}{2}c-y)^{\frac{1}{2}}(\tfrac{1}{2}a-x)\quad\ldots\ldots\ldots\ldots\ldots(6);$$

also the limits are as given below, and the number of ellipses is expressed by $\frac{1}{8}ac\sin B$; hence the average area of the ellipses is

$$\frac{8}{ac\sin B}\int_0^{\frac{1}{2}c}d\,(y\sin B)\int_{(\frac{1}{2}-\frac{y}{c})a}^{\frac{1}{2}a}(6)\,dx=$$

$$-\frac{16\pi}{a^2c^2}\int_0^{\frac{1}{2}c}(c-2y)^{\frac{1}{2}}\,dy\int_{\left(\frac{ay}{c}\right)^{\frac{1}{2}}}^{\prime}(ay-cu^2)^{\frac{1}{2}}\,2u^2du,\text{ where } u^2=\tfrac{1}{2}a-x,$$

$$=\frac{2\pi^2}{c^{\frac{7}{2}}}\int_0^{\frac{1}{2}c}(c-2y)^{\frac{1}{2}}\,y^2dy=\frac{4\pi^2}{105}\text{ of the triangle ABC.}$$

3. The area of the triangle BDF in parts of that of the triangle ABC is

$$\frac{\text{BD . BF}}{ac}=\alpha\beta=\left\{1+\frac{c\,(l-\frac{1}{2}a)}{am}\right\}\left\{1+\frac{a\,(m-\frac{1}{2}c)}{cl}\right\}\ldots\ldots\ldots\ldots(7);$$

hence the average area of the triangle BDF is

$$\frac{8}{ac\sin\text{B}}\int_0^{\frac{1}{2}c}d\,(m\sin\text{B})\int_{\left(\frac{1}{2}-\frac{m}{c}\right)a}^{\frac{1}{2}a}(7)\,dl$$

$$=-\frac{1}{2}-\frac{8}{c^2}\int_0^{\frac{1}{2}c}\frac{(m-\frac{1}{2}c)^2}{m}\log\left(1-\frac{2m}{c}\right)dm$$

$$=-3-\left\{\frac{4}{c^2}(m^2-2cm+\tfrac{3}{2}c^2)\log\left(1-\frac{2m}{c}\right)\right\}_0^{\frac{1}{2}c}+2\int_0^{\frac{1}{2}c}\frac{1}{m}\log\left(1-\frac{2m}{c}\right)dm$$

$$=-3+\frac{\pi^2}{3}.$$

The same holds good of the average area of each of the triangles CED, AFE; therefore the average area of the triangle DEF is

$$1-3\left(-3-\frac{\pi^2}{3}\right)\quad 10-\pi^2.$$

2590. (Proposed by Professor CAYLEY.)—It is required to verify Professor Kummer's theorem that "if a quartic surface is such that every section by a plane through a certain fixed point is a pair of conics, the surface is a pair of quadric surfaces (except only in the case where it is a quartic cone having its vertex at the fixed point)."

Solution by the PROPOSER.

The theorem may be more generally stated as follows; if a surface is such that every section through a certain fixed point (is or) includes a proper conic, then the surface (is or) includes a proper quadric surface. In order to the demonstration, I premise the following *Lemma*:—If a surface is such that every section through a certain fixed line includes a conic, then the line meets each of these conics in the same two points.

In fact, if the line meet the surface in any n points, then it is clear that each of the conics will meet the line in some two of these n points; and as

the plane of the section passes continuously from any one to any other position, the two points of intersection with the conic cannot pass abruptly from being some two to being some other two of the n points, that is, they are always the same two points.

Consider now a surface which is such that every section through a fixed point O includes a conic; and consider three lines xx', yy', zz' meeting in the point O. Let the conics in the planes yz, zx, xy be A, B, C respectively; then since the conics through the line xx' all pass through the same two points, and since B, C are two of these conics, B and C meet xx' in the same two points X, X'; similarly C and A meet yy' in the same two points Y, Y'; and A, B meet zz' in the same two points Z, Z'; that is, we have the conics A, B, C intersecting

B, C in the two points X, X',
C, A „ „ Y, Y',
A, B „ „ Z, Z;

hence taking on the conics A, B, C the points α, β, γ respectively, and drawing a quadric surface Σ through the nine points X, X, Y, Y,' Z, Z,' α, β, γ, this meets the conic A in the five points Y, Y', Z, Z,' α; that is, it passes through the conic A, and similarly it passes through the conic B, and through the conic C.

Consider how any plane whatever through O intersecting the conics A, B, C in the points L and L', M and M', N and N' respectively; the section of the quadric surface Σ by the plane in question is a conic through the six points L, L', M, M', N, N'. But the section of the surface includes a conic through these same six points, and which is consequently the same conic; in fact, the section of the surface by the plane in question includes a conic, and since every section through the line LL' includes a conic through the same two points, and one of these conics is the conic A which passes through the points L and L', the conic in question passes through the points L and L'; and similarly it passes through the points M and M, and through the points N and N'. That is, for any plane whatever through O, the section of the surface includes the conic which is the section of the quadric surface Σ, and the surface thus includes as part of itself the quartic surface Σ.

The foregoing demonstration ceases, however, to be applicable if O is a point on the surface, and the conic included in the section through O is always a conic passing through the point O. In the case where O is a non-singular point of the surface (that is, where there is at O a unique tangent plane) a like demonstration applies. Take through O a section, and let this include the conic A; on A take any point O' and through OO' a section including the conic B; we have thus the conics A, B intersecting in the points O, O'. Take through O any plane meeting the conics A, B in the points X, Y respectively—the section by this plane includes a conic C passing through the points O, X, Y; and each of the conics A, B, C touches at O the same plane, viz., the tangent plane of the surface. Hence, taking on the conic A the point α, consecutive to O, and any other point α'; on the conic B the point β, consecutive to O, and any other point β'; and on the conic C a point γ'; we may, through the nine points O, α, β, O', α', β', X, Y, γ' describe a quadric surface Σ; this will touch at O the tangent plane of the surface, that is, it will touch the conic C, or (what is the same thing) pass through a point γ of this conic consecutive to O. Hence the quadric surface meets the conic A in the five points O, O', α, α', X, that is, it entirely contains the conic A; similarly it meets the conic B in the five points O, O', B, B', Y, that is, it entirely contains the conic B; and it meets the conic C in the five points O, γ, X, Y, γ', that is, it entirely con-

tains this conic. And it may then be shown as before that the surface will include the quadric surface Σ. But there still remains for consideration the case where O is a conical point on the surface, and I do not at present see how this is to be treated.

I remark that, taking three lines xx', yy', zz' which meet in a point O, then if a surface be such that every section through xx' includes a conic, every section through yy' includes a conic, and every section through zz' includes a conic; and if besides the two points, say X, X′, on the conics through the line xx' are ordinary points on the surface, then the surface will include a quadric surface. In fact, if the surface has at each of the points X, X′ an ordinary tangent plane, then the conics through xx', and (as conics of the series) the two conics B, C all of them touch the two tangent planes; hence, constructing as before the quadric surface Σ, this also touches the two tangent planes: and taking through xx' a plane meeting the conic A in the points L, L′, the section of the surface includes a conic which touches the section of the quadric surface Σ at the points X, X′, and besides meets it in the points L, L′; such conic coincides therefore with the section of the quadric surface Σ; that is, every section of the surface through the line xx' includes the conic which is the section of the quadric surface Σ; and the surface thus includes as part of itself the quadric surface Σ.

2592. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a finite straight line be marked at random in $p-1$ points and broken up at those points, (1) the chance that none of the pieces is less than one-mth of the whole ($m>p$) is $\left(1-\frac{p}{m}\right)^{p-1}$; (2) the chance that none of the pieces is greater than one-nth of the whole ($n<p$) is

$$\left(\frac{p}{n}-1\right)^{p-1}-p\left(\frac{p-1}{n}-1\right)^{p-1}+\frac{p(p-1)}{2}\left(\frac{p-2}{n}-1\right)^{p-1}-\ldots\ldots$$

to r terms, r being the integer next $>p-n$; (3) the chance that none of the pieces is less than one-mth and none greater than one-nth is

$$\left(1-\frac{p}{m}\right)^{p-1}-p\left(1-\frac{p-1}{m}-\frac{1}{n}\right)^{p-1}+\frac{p(p-1)}{2}\left(1-\frac{p-2}{m}-\frac{2}{n}\right)^{p-1}-\ldots$$

to r terms, where r is the integer next $>n\frac{m-p}{m-n}$; provided that $\frac{1}{m}+\frac{p-1}{n}>1$ and $\frac{1}{n}+\frac{p-1}{m}<1$, otherwise one of the chances includes the other and has been already determined.

Solution by Professor WHITWORTH.

1. The number of ways in which mk *indifferent* things can be distributed into p *different* parcels, so that no parcel contain less than k, is (*Messenger of Mathematics*, No. XV., p. 155) $\frac{\lfloor (m-p)k+p-1}{\lfloor (m-p)k\,.\lfloor p-1}$. But the number of ways in which the distribution can be made without limitation is $\frac{\lfloor mk-1}{\lfloor mk-p\,\lfloor p-1}$. Hence if the distribution be made at random, the chance

that no parcel contain less than k is $\dfrac{\lfloor\underline{(m-p)k+p-1}\;\lfloor\underline{mk-p}}{\lfloor\underline{(m-p)k}\;\lfloor\underline{mk-1}}$. Now, suppose k be infinite, then the result becomes $\dfrac{(m-p)^{p-1}}{m^{p-1}}$ or $\left(1-\dfrac{p}{m}\right)^{p-1}$ which is therefore the chance that if an infinite series of elements be divided into p parts in order, no part will be less than one-mth of the whole.

2. The number of ways in which nk *indifferent* things can be distributed into p *different* parcels, no parcel containing more than k, is the coefficient of x^{nk} in the expansion of

$$(x+x^2+x^3+\ldots+x^k)^p = \text{co. of } x^{nk-p} \text{ in } (1-x^k)^p(1-x)^{-p}$$

$$=\frac{\lfloor\underline{nk-1}}{\lfloor\underline{nk-p}\;\lfloor\underline{p-1}}-\frac{p}{1}\cdot\frac{\lfloor\underline{(n-1)k-1}}{\lfloor\underline{(n-1)k-p}\;\lfloor\underline{p-1}}+\frac{p(p-1)}{1.2}\cdot\frac{\lfloor\underline{(n-2)k-1}}{\lfloor\underline{(n-2)k-p}\;\lfloor\underline{p-1}}-\&c.,$$

the number of terms being the integral part of n.

But the number of ways in which the parcels can be formed without limitation is $\dfrac{\lfloor\underline{nk-1}}{\lfloor\underline{nk-p}\;\lfloor\underline{p-1}}$. (*Messenger of Mathematics*, as above)

Hence, if the parcels be formed at random, the chance that none is greater than k is

$$1-\frac{p}{1}\cdot\frac{\lfloor\underline{nk-p}\;\lfloor\underline{(n-1)k-1}}{\lfloor\underline{nk-1}\;\lfloor\underline{(n-1)k-p}}+\frac{p(p-1)}{1.2}\cdot\frac{\lfloor\underline{nk-p}\;\lfloor\underline{(n-2)k-1}}{\lfloor\underline{nk-1}\;\lfloor\underline{(n-2)k-p}}-\&c.$$

Now suppose k infinite, then the result becomes

$$1-\frac{p}{1}\left(1-\frac{1}{n}\right)^{p-1}+\frac{p(p-1)}{1.2}\left(1-\frac{2}{n}\right)^{p-1}-\frac{p(p-1)(p-2)}{1.2.3}\left(1-\frac{3}{n}\right)^{p-1}+\&c.,$$

(the number of terms being n if integral, or the next greater integer), which is therefore the chance that if an infinite series of elements be divided into p parts in order, no part will be greater than one-nth of the whole.

This is the form in which Mr. Wolstenholme proposed the result in the Mathematical Tripos Examination, Jan. 15, 1868.

3. The number of ways in which mnk *indifferent* things can be distributed into p parcels, so that no parcel contain less than nk nor more than mk, is found by supposing $nk-1$ things placed in each of the p parcels, and then considering how the remaining $(m-p)nk+p$ things can be further distributed into the p parcels without putting more than $(m-n)k+1$ into each parcel.

Therefore by the preceding case the result is the coefficient of $x^{(m-p)nk+p}$ in the expansion of $(x+x^2+x^3+\ldots+x^{(m-n)k+1})^p$

$$=\frac{\lfloor\underline{(m-p)nk+p-1}}{\lfloor\underline{(m-p)nk}\;\lfloor\underline{p-1}}-\frac{p}{1}\cdot\frac{\lfloor\underline{(m-p)nk-(m-n)k+p-2}}{\lfloor\underline{(m-p)nk-(m-n)k-1}\;\lfloor\underline{p-1}}+\&c.,$$

the number of terms being $\dfrac{(m-p)nk+p}{(m-n)k+1}$ or the next greater integer.

But the number of ways in which the parcels can be formed without limitation is

$$\frac{\underline{|mnk-1}}{\underline{|mnk-p}\;\underline{|p-1}}.$$

Hence if the parcels be formed at random, the chance that none is less than nk or greater than mk is

$$\frac{\underline{|(m-p)nk+p-1}\;\underline{|mnk-p}}{\underline{|(m-p)nk}\;\underline{|mnk-1}}-\frac{p}{1}\cdot\frac{\underline{|(m-p)nk-(m-n)k+p-2}\;\underline{|mnk-p}}{\underline{|(m-p)nk-(m-n)k-1}\;\underline{|mnk-1}}$$
$$+\,\&c.$$

Now suppose k infinite: then the result becomes

$$\left\{\frac{(m-p)n}{mn}\right\}^{p-1}-\frac{p}{1}\left\{\frac{(m-p)n-(m-n)}{mn}\right\}^{p-1}$$
$$+\frac{p\,.\,(p-1)}{1\,.\,2}\left\{\frac{(m-p)n-2(m-n)}{mn}\right\}^{p-1}-\&c.,$$

or

$$\left\{1-\frac{p}{m}\right\}^{p-1}-p\left\{1-\frac{p-1}{m}-\frac{1}{n}\right\}^{p-1}+\&c.,$$

$\left(\text{the number of terms being the integral part of } \dfrac{(m-p)n}{m-n},\right)$ which is therefore the chance that, if an infinite series of elements be divided into p parts in order, no part will be less than one-mth or greater than one-nth of the whole.

ON THE INTEGRATION OF DIFFERENTIAL EQUATIONS.

By Chief Justice Cockle.

1. Theorem I. If the coefficients of the two linear differential equations of the second order

$$\frac{d^2y}{dx^2}+2q\frac{dy}{dx}+ry=0,\qquad \frac{d^2Y}{dx^2}+2Q\frac{dY}{dx}+RY=0\;\ldots\ldots\ldots(1,2),$$

are connected by the relation $q^2+\dfrac{dq}{dx}-r=Q^2+\dfrac{dQ}{dx}-R\;\ldots\ldots\ldots\ldots(3),$

then also are the dependent variables y and Y connected by the relation

$$Ce^{\int(q-Q)\,dx}\,y=Y\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

For, in (2) substitute uy for Y and divide the result by u. We find

$$\frac{d^2y}{dx^2}+2\left\{\frac{1}{u}\left(\frac{du}{dx}\right)+Q\right\}\frac{dy}{dx}+\left\{\frac{1}{u}\left(\frac{d^2u}{dx^2}\right)+2Q\frac{1}{u}\left(\frac{du}{dx}\right)+R\right\}y=0\;\ldots.(5).$$

Now, in order that (5) may be identical with (1), we have the following conditions:

$$\frac{1}{u}\cdot\frac{du}{dx}+Q=q,\qquad \frac{1}{u}\cdot\frac{d^2u}{dx^2}+2Q\frac{1}{u}\cdot\frac{du}{dx}+R=r\;\ldots\ldots\ldots(6,7).$$

Differentiate (6) with respect to the independent variable x, and subtract the result from (7). We have

$$\frac{1}{u^2}\left(\frac{du}{dx}\right)^2+2Q\frac{1}{u}\cdot\frac{du}{dx}+R-\frac{dQ}{dx}=r-\frac{dq}{dx}\ldots\ldots\ldots\ldots(8).$$

Next, in (8), substitute for $\frac{1}{u}\cdot\frac{du}{dx}$ its value $(q-Q)$ deduced from (6). Then (8) becomes

$$(q-Q)^2+2Q(q-Q)+R-\frac{dQ}{dx}=r-\frac{dq}{dx}\ \ldots\ldots\ldots\ldots(9),$$

or

$$q^2-Q^2+R-r+\frac{dq}{dx}-\frac{dQ}{dx}=0\ \ldots\ldots\ldots\ldots(10),$$

which last equation is equivalent to (3), and the theorem is demonstrated, for if u be determined from (6) the relation $uy=Y$ conducts us to (4).

2. THEOREM II. The binordinary $\frac{d^2y}{dx^2}-\left\{\frac{1}{2r}\cdot\frac{dr}{dx}+a\sqrt{r}\right\}\frac{dy}{dx}+ry=0\ldots(11)$ is soluble whatever function of x the quantity r may be, for it is transformable into an equation with constant coefficients.

This will most readily appear by changing the independent variable from x to t, which latter quantity is determined by means of

$$\int\sqrt{r}\,.\,dx=Ct+C_2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

C and C_2 being constants. If the latter constant be supposed to vanish, the transformed equation will be

$$\frac{d^2y}{dt^2}-aC\frac{dy}{dt}+C^2y=0\ \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

whereof all the coefficients are constant. The relation

$$\left(\frac{d}{dx}-\frac{1}{2r}\cdot\frac{dr}{dx}+\beta\sqrt{r}\right)\left(\frac{d}{dx}+\gamma\sqrt{r}\right)y=0\ \ldots\ldots(14),$$

wherein β and γ are determined from

$$\beta+\gamma=a,\quad \beta\gamma=1\ \ldots\ldots\ldots\ldots\ldots\ldots(15,16),$$

is a symbolical decomposition, and consequently a means of solution, of (11). This decomposition, and in general all such decompositions, will be followed without much difficulty by operating first on y with the right hand factor and then on the result with the left hand factor, and so on for any number of symbolical factors. If we divide (11) by r, the result, viz.,

$$\frac{1}{r}\cdot\frac{d^2y}{dx^2}-\left\{\frac{1}{2r^2}\cdot\frac{dr}{dx}+a\frac{1}{\sqrt{r}}\right\}\frac{dy}{dx}+y=0\ \ldots\ldots\ldots\ldots(17),$$

is symbolically decomposable into

$$\left(\frac{1}{\sqrt{r}}\cdot\frac{d}{dx}+\beta\right)\left(\frac{1}{\sqrt{r}}\cdot\frac{d}{dx}+\gamma\right)y=0\ldots\ldots\ldots\ldots(18),$$

where β and γ are determined from (15, 16) as before. Hence we are readily led to (12). For if

$$\frac{1}{\sqrt{r}}\cdot\frac{dy}{dx}=\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}\ \ldots\ldots\ldots\ldots\ldots(19),$$

we have

$$\frac{1}{\sqrt{r}}=\frac{dx}{dt}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

which is easily reducible to a case of (12).

3. There are two other forms, soluble for all values of r, and somewhat resembling (11), viz.,

$$\frac{d^2y}{dx^2}-\left(\frac{1}{r}\cdot\frac{dr}{dx}+ar+\frac{1}{a}\right)\frac{dy}{dx}+ry=0\ \ldots\ldots\ldots\ldots(21),$$

and

$$\frac{d^2y}{dx^2}-\left(\frac{1}{r}\cdot\frac{dr}{dx}+ar+b\right)\frac{dy}{dx}+\frac{b}{r}\cdot\frac{dr}{dx}\cdot y=0\ \ldots\ldots(22).$$

The symbolical decomposition of (21) is

$$\left(\frac{d}{dx} - \frac{1}{r}\cdot\frac{dr}{dx} - \frac{1}{a}\right)\left(\frac{d}{dx} - ar\right)y = 0 \ldots\ldots\ldots\ldots (23),$$

and that of (22) is $\left(\frac{d}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}\right)\left(\frac{d}{dx} - ar - b\right)y = 0 \ldots\ldots\ldots (24).$

4. If we develope the symbolical equation

$$\left\{(b+cx)\frac{d}{dx} + \frac{r}{a}\right\}\left\{(b+cx)\frac{d}{dx} + a\right\}y = 0 \ldots\ldots\ldots\ldots (25),$$

we obtain the result $(b+cx)^2\frac{d^2y}{dx^2} + \left(\frac{r}{a} + a + c\right)(b+cx)\frac{dy}{dx} + ry = 0 \ldots(26),$

and this last equation is therefore soluble for any value whatever of r. An easy illustration of symbolical decomposition is afforded by the equation

$$\left(\frac{d}{dx} + b + ax\right)\left(\frac{d}{dx} + b - ax\right)y = 0 \ldots\ldots\ldots\ldots (27),$$

which, developed, becomes

$$\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + (b^2 - a - a^2x^2)y = 0 \ldots\ldots\ldots\ldots (28).$$

5. *Scholium.*—If we call the functions on either side of equation (3) criticoids, Theorem I. indicates that if an equation having a certain criticoid be soluble, then all equations having the same criticoid are soluble.

6. Let us apply portions of the foregoing discussion to some of the equations of Riccati. And, first, let

$$\frac{d^2y}{dx^2} + \frac{a}{x^2}.y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29),$$

an equation which by exponential substitution (*i.e.* by taking $y = e^{\int u dx}$) may be transformed from its present binomial form to the trinomial form considered by Riccati. Now, comparing (29) with (11) we see that the former equation may be solved by change of the independent variable, and (12) suggests the assumption

$$t = \log x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

which, being made, changes (29) into

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + ay = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

an equation with constant coefficients.

7. Next let $\frac{d^2y}{dx^2} + \frac{a}{x^4}.y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$

In this case, still conforming with (12), let $t = \frac{1}{x} \ldots\ldots\ldots\ldots\ldots (33),$

then $\frac{dx}{dt} = -\frac{1}{t^2}$ and $\frac{d^2x}{dt^2} = \frac{2}{t^3}$. Hence (32) becomes

$$\frac{d^2y}{dt^2} + \frac{2}{t}\cdot\frac{dy}{dt} + ay = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34),$$

and the criticoid of this equation is the same as that of

$$\frac{d^2y}{dt^2} + ay = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

an equation with constant coefficients. Consequently, by Theorem I., equation (34) is soluble, and the solution of (32) is in substance reduced to that of (35).

8. If v be an integral of (34) then tv is an integral of (35), and if w be an integral of (35) then $\frac{w}{t}$ is an integral of (34). And it may be convenient to remark here that the solution of

$$\frac{d^2y}{dx^2}+ry=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

depends upon that of $\frac{d^2z}{dx^2}-\frac{1}{r}\cdot\frac{dr}{dx}\cdot\frac{dz}{dx}+rz=0 \ \ldots\ldots\ldots\ldots\ldots\ldots (37)$

and *vice versâ*. For if we divide (36) by r, differentiate the quotient, multiply the result into r, and then replace $\frac{dy}{dx}$ by z, the final equation thus obtained will be (37).

9. Next consider the equation $\frac{d^2y}{dx^2}+\frac{a}{\sqrt[3]{x^4}}y=0 \ \ldots\ldots\ldots\ldots\ldots\ldots (38)$,

from which we derive

$$\frac{d^2z}{dx}+\frac{4}{3}\cdot\frac{1}{x}\cdot\frac{dz}{dx}+\frac{a}{\sqrt[3]{x^4}}\cdot z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

in the same way that (37) is derived from (36).

Let
$$x=\frac{1}{t^3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

and consequently
$$\frac{dx}{dt}=-\frac{3}{t^4},\ \frac{d^2x}{dt^2}=\frac{12}{t^5};$$

then (39) may be transformed into

$$\frac{d^2z}{dt^2}+\left\{\frac{4}{3}t^3\left(-\frac{3}{t^4}\right)+\frac{4}{t}\right\}\frac{dy}{dt}+at^4\left(\frac{3^2}{t^8}\right)y=0 \quad \ldots\ldots\ldots (41),$$

or, reducing, into
$$\frac{d^2z}{dt^2}+\frac{9a}{t^4}\cdot y=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42),$$

o that the solution of (38) is thus made to depend upon that of (32).

10. Another transformation may be applied to (39). Let

$$x=\frac{\epsilon^{-3ct}}{-3c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43),$$

and therefore
$$\frac{dx}{dt}=\epsilon^{-3ct},\ \frac{d^2x}{dt^2}=-3c\epsilon^{-3ct}$$

then (39) may be transformed into

$$\frac{d^2z}{dt^2}+\left\{\tfrac{4}{3}(-3c)+3c\right\}\frac{dz}{dt}+a\,(-3c)^{\frac{4}{3}}\,\epsilon^{-2ct}\,z=0\ \ldots\ldots\ldots (44),$$

which, if we put
$$a\sqrt[3]{\left\{(-3c)^4\right\}}=A,$$

becomes
$$\frac{d^2z}{dt^2}+c\frac{dz}{dt}+A\epsilon^{-2ct}\cdot z=0\ldots\ldots\ldots\ldots\ldots\ldots (45).$$

11. Since (42) is soluble, (45) is soluble, as also is

$$\frac{d^2z}{dt^2}+\left\{Ae^{-2ct}-\frac{c^2}{4}\right\}z=0\ldots\ldots\ldots\ldots\ldots\ldots(46),$$

which has the same criticoid as (45) and in which A and c are any constants whatever. And it may be convenient to remark here that whenever c is constant and

$$\frac{d^2y}{dx^2}+c\frac{dy}{dx}+ry=0\ldots\ldots\ldots\ldots\ldots\ldots(47)$$

soluble, then also the equation

$$\frac{d^2y}{dx^2}-c\frac{dy}{dx}+ry=0\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

is soluble, for (47) and (48) have the same criticoid.

12. Theorems I. and II. are of considerable generality, and it may be interesting to compare Boole's *Differential Equations*, 2nd ed., p. 411, Ex. 11; p. 459, Ex. 14; and pp. 428–429, Art. 6. With Ex. 11 of p. 411, just cited, may be compared the equation

$$\frac{d^2y}{dx^2}+(b+X)\frac{dy}{dx}+bXy=0\ \ldots\ldots\ldots\ldots\ldots(9),$$

in which X is any function whatever of x. This equation is deducible from (22) by making a vanish and performing an easy substitution. If in (37) we change the independent variable from x to t, and so make the middle term of (37) a constant, say c, we find

$$\int r\,dx=-\frac{1}{c}\,.\,\epsilon^{-ct}\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

so that if $r=x^m$ we have $\quad x=\left(-\frac{m+1}{c}\right)^{\frac{1}{m+1}}.\,\epsilon^{-\frac{ct}{m+1}}\ldots\ldots(51)$

and $\quad r\left(\frac{dx}{dt}\right)^2=\frac{1}{r}\,.\left(r\frac{dx}{dt}\right)^2$

$$=\epsilon^{-2ct}.\left(-\frac{m+1}{c}\right)^{-\frac{m}{m+1}}.\,\epsilon^{\frac{mct}{m+1}}=\left(-\frac{m+1}{c}\right)^{-\frac{m}{m+1}}.\,\epsilon^{-\left(\frac{m+2}{m+1}\right)ct}\ldots.(52).$$

Now, applying theorem II., we know that, A being any constant,

$$\frac{d^2z}{dt^2}+c\frac{dz}{dt}+A\epsilon^{-\left(\frac{m+2}{m+1}\right)ct}z=0\ \ldots\ldots\ldots\ldots\ldots\ldots(53)$$

will be soluble by a change of the independent variable provided that

$$c=\frac{1}{2}\left(\frac{m+2}{m+1}\right)c,\ \text{or}\ m=0\ldots\ldots\ldots\ldots\ldots\ldots(54).$$

Again, as in Art. 11, we infer that when (53) is soluble, then also

$$\frac{d^2z}{dt^2}-c\frac{dz}{dt}+Ae^{-\left(\frac{m+2}{m+1}\right)ct}z=0\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

But, by Theorem II., (55) is soluble when

$$-c=\left(\frac{m+2}{m+1}\right)\frac{c}{2},\ \text{or}\ m=-\frac{4}{3}\ldots\ldots\ldots\ldots\ldots\ldots(56),$$

and we are substantially conducted to a solution of (38).

With this last value of m, (55) becomes an equation with the same criticoid as (46). Compare Mr. Hargreave's remarks at p. 258 of Vol. VII. of the *Quarterly Journal;* also mine at pp. 243, 244, of Vol. I. of the *Messenger.* Mr. Hargreave treats a form corresponding to (29) by a process of inspection or substitution, and a form corresponding to (46) by obtaining a general value from a particular solution. My discussion connects the latter form with the theory of coresolvents. I may here add that, if we change t in (46) into $-t$, that equation becomes

$$\frac{d^2z}{dt^2}+\left\{Ae^{2ct}-\frac{c^2}{4}\right\}z=0\ldots\ldots\ldots\ldots\ldots\ldots(57),$$

and the origin of the double value of m in (54) and (56) is thus elucidated.

13. By Theorem II., we know that

$$\frac{d^2y}{dx^2}-\frac{n}{2}\cdot\frac{1}{x}\cdot\frac{dy}{dx}+ax^ny=0\ \ldots\ldots\ldots\ldots\ldots\ldots(58)$$

is soluble whatever n may be. If $n=-4$ then (58) becomes

$$\frac{d^2y}{dx^2}+\frac{2}{x}\cdot\frac{dy}{dx}+\frac{a}{x^4}\cdot y=0\ldots\ldots\ldots\ldots\ldots\ldots(59),$$

which has the same criticoid as (32), the solubility of which is thus manifested. In general, the criticoid of (58) is

$$ax^n-\frac{n(n+4)}{16}\cdot\frac{1}{x^2}$$

and the solution of (58) may be made to depend upon that of

$$\frac{d^2y}{dx^2}+\left\{ax^n-\frac{n(n+4)}{16}\cdot\frac{1}{x^2}\right\}y=0\ldots\ldots\ldots\ldots(60),$$

and this again, by exponential substitution, upon that of

$$\frac{dz}{dx}+z^2+ax_n-\frac{n(n+4)}{16}\cdot\frac{1}{x^2}=0\ \ldots\ldots\ldots\ldots(61).$$

Assume $\qquad z=\lambda x^{\frac{n}{2}}+\frac{\mu}{x}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$

and substitute in (61). We have, λ and μ being constant,

$$\frac{\lambda n}{2}\cdot x^{\frac{n}{2}-1}-\frac{\mu}{x^2}+(\lambda^2+a)\,x^n+2\lambda\mu x^{\frac{n}{2}-1}+\frac{16\mu^2-n\,(n+4)}{16x^2}=0,$$

or $\qquad (\lambda^2+a)\,x^n+\lambda\left(\frac{n}{2}+2\mu\right)x^{\frac{n}{2}-1}+\frac{16\mu\,(\mu-1)-n\,(n+4)}{16x^2}=0\ldots.(63),$

and (63) is satisfied if we make

$$\lambda^2+a=0\text{ and }n+4\mu=0\ldots\ldots\ldots\ldots\ldots\ldots(64, 65).$$

We might of course have solved (58) at once by changing the independent variable.

14. In extension of Ex. 27 of p. 236 of Boole (op. cit.), it may be observed that $x^{n-1}\cdot\dfrac{d^ny}{dx^n}$ is an exact differential coefficient. For

$$\int x^{n-1}\frac{d^ny}{dx^n}\,dx=x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}-(n-1)\int x^{n-2}\frac{d^{n-1}y}{dx^{n-1}}\,dx\ \ldots\ldots(66).$$

15. In certain earlier steps the theory of coresolvents may be brought into relation with the theory of the comparison of transcendents. Thus, differentiating the equation $y^n + x^n = C$ .... (67),

we have, after dividing by n, $y^{n-1}dy + x^{n-1}dx = 0$ .... (68),

or, by (67), $(C-x^n)^{\frac{n-1}{n}} dy + (C-y^n)^{\frac{n-1}{n}} dx = 0$ .... (69),

or, after a division, $\dfrac{dy}{(C-y^n)^{\frac{n-1}{n}}} + \dfrac{dx}{(C-x^n)^{\frac{n-1}{n}}} = 0$ .... (70),

of which the primitive is (67). When $n=2$, then (70) becomes

$$\frac{dy}{\sqrt{C-y^2}} + \frac{dx}{\sqrt{C-x^2}} = 0 \text{ .... (71),}$$

a familiar form. If $n=\frac{3}{2}$, then (70) becomes

$$\frac{dy}{\sqrt[3]{C-y^{\frac{3}{2}}}} + \frac{dx}{\sqrt[3]{C-x^{\frac{3}{2}}}} = 0 \text{ .... (72).}$$

16. If we factorially differentiate (58), that is, put it under the form

$$\frac{1}{x^n} \cdot \frac{d^2y}{dx^2} - \frac{n}{2} \cdot \frac{1}{x^{n+1}} \cdot \frac{dy}{dx} + ay = 0 \text{ .... (73),}$$

and differentiate, we have, replacing $\dfrac{dy}{dx}$ by z,

$$\frac{1}{x^n} \cdot \frac{d^2z}{dx^2} - \frac{3n}{2} \cdot \frac{1}{x^{n+1}} \cdot \frac{dz}{dx} + \left\{a + \frac{n(n+1)}{2} \cdot \frac{1}{x^{n+2}}\right\} z = 0 \text{ .... (73);}$$

or, multiplying into x^n,

$$\frac{d^2z}{dx^2} - \frac{3n}{2x} \cdot \frac{dz}{dx} + \left\{ax^n + \frac{n(n+1)}{2} \cdot \frac{1}{x^2}\right\} z = 0 \text{ .... (74),}$$

and the criticoid of this equation is

$$ax^n + \left\{\frac{n(n+1)}{2} - \frac{9n^2}{16} - \frac{3n}{4}\right\} \frac{1}{x^2},$$

which, reduced, becomes $\quad ax^n - \dfrac{n^2+4n}{16},$

which is equal to the criticoid of (58), and the criticoid has been unchanged by the differentiation.

17. Equation (58) is soluble by change of the independent variable. In ordinary cases we may find equations, the solutions of which depend upon that of a given equation, but which have not the same criticoids. Thus the equations

$$\frac{d^2y}{dx^2} + a^2x^2y = 0, \quad \frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} + a^2x^2y = 0 \text{ ...... (74, 75)}$$

have the same criticoid, viz. a^2x^2. But if we factorially differentiate (74), we have, replacing $\dfrac{dy}{dx}$ by z, and proceeding in other respects as in Art. 16 for our result

$$\frac{d^2z}{dx^2} - \frac{2}{x} \cdot \frac{dz}{dx} + a^2x^2z = 0 \text{ .... (76),}$$

whereof the criticoid is $\qquad a^2x^2 - \dfrac{2}{x^2};$

while if we factorially differentiate (75), and proceed as in the last case, we are conducted to

$$\frac{d^2z}{dx^2} + \left(a^2x^2 - \frac{6}{x^2}\right)z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (77),$$

whereof the criticoid is $\qquad a^2x^2 - \dfrac{6}{x^2}.$

Although invariable under factorial substitution, the criticoid is not necessarily so under differentiation or change of the independent variable.

2583. (Proposed by J. J. WALKER, M.A.)—Prove that the expression

$$\left\{(a-b)^2(c-d)^2+(a-c)^2(b-d)^2+(a-d)^2(b-c)^2\right\}^3$$
$$-8\left\{(a-b)^3(b-c)^3(c-d)^3(d-a)^3+(a-b)^3(b-d)^3(d-c)^3(c-a)^3 + (c-b)^3(b-d)^3(d-a)^3(a-c)^3\right\}$$
$$-24(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$$

vanishes identically.

Solution by W. H. LAVERTY; S. BILLS; *the* PROPOSER; *and others.*

Let $\qquad (a-c)^2(b-d)^2+(a-d)^2(b-c)^2 = \mathrm{P},$

and $\qquad (a-c)(b-d)(a-d)(b-c) = \mathrm{Q},$

then $\qquad \mathrm{P}-2\mathrm{Q} = (a-b)^2(c-d)^2 = \mathrm{R}$ (suppose),

and $\qquad -(b-c)^3(a-d)^3+(b-d)^3(a-c)^3 = \mathrm{R}^{\frac{1}{2}}(\mathrm{P}+\mathrm{Q});$

hence the given expression assumes the form

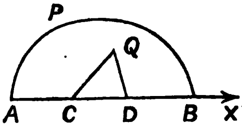
$$\left\{(\mathrm{P}-2\mathrm{Q})+\mathrm{P}\right\}^3 - 8\left\{\mathrm{R}^{\frac{3}{2}}\mathrm{R}^{\frac{1}{2}}(\mathrm{P}+\mathrm{Q})+\mathrm{Q}^3\right\} - 24(\mathrm{P}-2\mathrm{Q})\mathrm{Q}^2,$$

which, divided by 8, and simplified, becomes

$$= \mathrm{P}^3-3\mathrm{P}^2\mathrm{Q}+4\mathrm{Q}^3-\mathrm{R}^2(\mathrm{P}+\mathrm{Q}) = (\mathrm{P}-2\mathrm{Q})^2(\mathrm{P}+\mathrm{Q})-\mathrm{R}^2(\mathrm{P}+\mathrm{Q}) = 0.$$

2477. (Proposed by T. SAVAGE, M.A.)—If there be described any curve APB upon the chord AB, the average area of all the triangles, formed by joining a point Q taken at random within the area APB and two fixed points C, D in the line AB, is equal to the area of the triangle formed by joining C, D and the centre of gravity of the area APB.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.



Since the curve is all on one side of AB, it is obvious that the average area (Δ say) of the triangle CQD is

$$\Delta = \tfrac{1}{2}\mathrm{CD}\iint y\,dx\,dy \div \iint dx\,dy$$

over the area APB, AB being the axis of x; that is, if (ξ,η) be the coordinates of the centre of gravity G of the area APB, we have $\Delta = \frac{1}{2}\mathrm{CD}\,.\,\eta$ = area of triangle CGD.

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